

THE RANDOM MATRIX REGIME OF MARONNA'S M-ESTIMATOR FOR OBSERVATIONS CORRUPTED BY ELLIPTICAL NOISES

MOHAMED-SLIM ALOUINI AND ABLA KAMMOUN

ABSTRACT. This article studies the behavior of the Maronna robust scatter estimator $\hat{C}_N \in \mathbb{C}^{N \times N}$ of a sequence of observations y_1, \dots, y_n which is composed of a K dimensional signal drown in a heavy tailed noise, i.e $y_i = A_N s_i + x_i$ where $A_N \in \mathbb{C}^{N \times K}$ and x_i is drawn from elliptical distribution. In particular, we prove that as the population dimension N , the number of observations n and the rank of A_N grow to infinity at the same pace and under some mild assumptions, the robust scatter matrix can be characterized by a random matrix \hat{S}_N that follows a standard random model. Our analysis can be very useful for many applications of the fields of statistical inference and signal processing.

1. INTRODUCTION

Estimation of covariance matrices is at the heart of the theory of multivariate statistical analysis [12]. Its importance can be seen from its broad range of applications including financial data analysis, statistical signal processing, and wireless communication. A natural way to estimate covariance matrices is represented by the sample covariance matrix. Given n observations y_1, \dots, y_n , of size N , independent, and identically distributed, (i.i.d) then the sample covariance matrix is given by $\frac{1}{n} \sum_{i=1}^n y_i y_i^*$. The popularity of the sample covariance matrix essentially comes from its low-complexity and the existence of a good understanding of its behaviour in two asymptotic regimes: n goes to infinty while N is fixed when N and n go to infinity with the same pace. Recent advances in the theory of large random matrices have made it clear that in the second asymptotic regime, the sample covariance matrix is no longer consistent. Conventional estimation methods that are based on the use of the sample covariance matrix are thus inefficient when the number of observations and their dimension become commensurate and large. Such scenario naturally arises in current array processing applications where the trend is to employ large antenna arrays. Based on a deep understanding of the behaviour of the sample covariance matrix, a new wave of detection methods [4, 3, 13] and subspace estimation techniques [11, 14, 18] has recently emerged. Although consistent, these methods are bound by the fact that they still fundamentally rely on the sample covariance matrix, their consistency being obtained by resorting to a deep analysis of its asymptotic behaviour. Nevertheless, the use of the sample covariance matrix can lead to poor performances, especially when observations are drawn from heavy tailed distributions or contain outliers. In such situations, the use of robust covariance estimators has been aknowledged as an efficient solution to combat the presence of outliers. Although references to robust techniques are traced back to the eighties with the works of Huber [9] and Maronna [10], the study of their performance has been often restricted to the conventional regime where the number of observations is too large as compared to their dimensions. It

was only recently that new tools have been developed in [6, 7, 8] which allow to analyse the behaviour of robust Maronna's scatter estimators. The main contributors are Couillet et al. who established that the robust scatter estimator can be well-approximated in the asymptotic regime by a random matrix that follows a standard random model. One of the key advantages of this result, is that it allows to bring back the asymptotic analysis of robust-scatters to that of an other random object for which an important load of results already exist.

Despite their high value, these works have been derived only for the case of pure noise observations. While the case of a low rank signal observations can be dealt with by resorting to easy adaptations of the approach of [8], handling high-rank signal observations is much more challenging. Building on the tools developed in [8], we propose in this work to analyse this difficult scenario. We show that in this case the adaption of the method in [8] is not immediate and necessitates the development of additional appropriate tools. Some of the required results that were of independent interest were submitted in an other work which can be found in [1].

Notations: In the remainder of this work, we shall denote $\lambda_1(X) \leq \dots \leq \lambda_N(X)$ the real eigenvalues of $n \times n$ Hermitian matrix X . The notation $\|\cdot\|$ will refer to the spectral norm of matrices and Euclidean norm for vectors, while $*$ will stand for the complex conjugate operator. The derivative of a differentiable function f will be denoted by f' .

2. ASSUMPTIONS AND MAIN RESULTS

We start by introducing the data model under study. We consider n sample vectors $y_1, \dots, y_n \in \mathbb{C}^N$ satisfying:

$$y_i = A_N s_i + x_i, i = 1, \dots, n,$$

where A_N is a $N \times K$ deterministic matrix and x_1, \dots, x_n are random vectors defined as:

$$x_i = \sqrt{\tau_i} w_i$$

with the scalars $\tau_1, \dots, \tau_n \in \mathbb{R}_+$. Let $\bar{N} = K + N$. We denote by $c_N = \frac{N}{n}$ and considers the following assumptions:

Assumption A-1. For each N , $c_N < 1$, $\bar{c}_N \geq 1$, and

$$c_- < \liminf_n c_N < \limsup_n c_N < c_+,$$

with $0 < c_- < c_+ < 1$.

This paper studies the asymptotic behaviour of the Maronna's M-robust scatter estimator in the regime of Assumption 1. We recall that the Maronna's M-robust estimator which we denote by \hat{C}_N is given by the unique solution in Z of the following equation:

$$Z = \frac{1}{n} \sum_{i=1}^n u \left(\frac{1}{N} y_i^* Z^{-1} y_i \right) y_i y_i^*. \quad (2.1)$$

where function $u(\cdot)$ satisfies the following properties:

- Assumption A-2.** i) Function $u(\cdot) : [0, \infty) \rightarrow [0, \infty)$ is non-negative continuous and non-increasing,
 ii) The function $\phi(\cdot) : x \mapsto xu(x)$ is increasing, bounded and continuously differentiable with $\lim_{x \rightarrow \infty} \phi(x) \triangleq \phi_\infty > 1$ and $\phi' > 0$.

iii) $\phi_\infty < c_+^{-1}$.

and the scalars τ_i are such that:

Assumption A-3. i) The random empirical measure $\nu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\tau_i}$ converges weakly to ν which satisfies $\int x \nu(x) = 1$,
 ii) There exists $\epsilon < 1 - \phi_\infty^{-1} < 1 - c_+$ and $m > 0$ such that for all large n a.s., $\nu([0, m]) < \epsilon$.

The conditions in Assumption 2 are the same as those considered in [8]. It is worth observing that Assumption 2-ii) is different from the one considered by Maronna in [10], in that ϕ is not allowed to be constant on any open interval. However, Assumption 2:-iii) is much more adapted to the high-dimensional regime than Assumption (D) p.53 of [10], which requires that $\phi_\infty > N$.

Assumption 3 is different from the original assumption in [8] as we assume here the weak convergence of the empirical measure ν_n . However, one can easily see by the Portmanteau lemma that Assumption 3 will bring about the same useful requirements, namely the a.s. tightness of $\{\nu_n\}_{n=1}^\infty$, i.e., for each $\eta > 0$, there exists $M > 0$ such that with probability one, $\nu_n([M, \infty)) < \eta$, along with the absence of a heavy mass concentrating close to zero ($\nu_n([0, m]) < \epsilon$ for n large enough a.s.).

The statistical hypothesis on y_1, \dots, y_n is detailed below:

Assumption A-4. i) $w_1, \dots, w_n \in \mathbb{C}^N$ are independent invariant complex zero-mean vectors with for each i , $\|w_i\|^2 = N$ and are independent of τ_1, \dots, τ_n ,
 ii) $s_i \sim \mathcal{CN}(0, I_K)$, $i = 1, \dots, n$ are independent standard Gaussian distributed vectors.
 iii) Define $B_N = A_N A_N^*$, then $\limsup \|B_N\| < \infty$ and $\liminf \frac{1}{N} \text{Tr } B_N > 0$.

In addition to the above assumptions, the following hypothesis might be required:

Assumption A-5. For each $a > b > 0$, a.s.,

$$\limsup_{t \rightarrow \infty} \frac{\limsup_n \nu_n(t, \infty)}{\phi(at) - \phi(bt)} = 0.$$

Theorem 2.1 (Uniqueness). Let Assumptions 1-4 hold true. Then, for all large n , (2.1) admits a unique solution \hat{C}_N . Moreover, \hat{C}_N is the limit of the sequence $Z^{(t)}$ given by:

$$Z^{(t+1)} = \frac{1}{n} \sum_{i=1}^n u \left(\frac{1}{N} y_i^* (Z^t)^{-1} y_i \right) y_i y_i^*,$$

where $Z^{(0)} \succeq 0$.

Theorem 2.2. Let Assumptions 1-5 hold. Let \hat{C}_N be given by Theorem 2.1 when uniquely defined. Then,

$$\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N = \frac{1}{n} \sum_{i=1}^n v(\delta_i) y_i y_i^*$$

and $\delta_1, \dots, \delta_n$ are the unique positive solutions in x_1, \dots, x_n to the following system of equations

$$x_i = \frac{1}{N} \text{Tr} (B_N + \tau_i I_N) \left(\frac{1}{n} \sum_{j=1}^n \frac{v(x_j)(B_N + \tau_j I_N)}{1 + c\psi(x_j)} \right), \quad (2.2)$$

with the functions $v : x \mapsto (u \circ g^{-1})(x)$, $\psi(\cdot) : x \mapsto xv(x)$ and $g(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}, x \mapsto x/(1 - c_N\phi(x))$.

Corollary 2.3. *Let Assumptions 1-5 hold true. Let \hat{C}_N be the solution of (2.1) when uniquely defined. Assume further that the empirical distribution F^{B_N} converges in distribution to F^B , a cumulative distribution function and $c_N \rightarrow c$. Set χ_∞ and γ_∞ the unique solutions to the following system of equations:*

$$\begin{aligned} \chi_\infty &= \int \frac{y}{\int_0^{+\infty} \frac{v(\chi_\infty + t\gamma_\infty)(y+t)}{1+c\psi(\chi_\infty+t\gamma_\infty)} \nu(dt)} F^B(dy) \\ \gamma_\infty &= \int \frac{1}{\int_0^{+\infty} \frac{v(\chi_\infty + t\gamma_\infty)(y+t)}{1+c\psi(\chi_\infty+t\gamma_\infty)} \nu(dt)} F^B(dy). \end{aligned}$$

Then,

$$\left\| \hat{C}_N - S_N \right\| \xrightarrow{\text{a.s.}} 0$$

where $S_N = \frac{1}{n} \sum_{i=1}^n v(\chi_\infty + \tau_i \gamma_\infty) y_i y_i^*$.

Proof. Let $\delta_1, \dots, \delta_n$ be the solution of the system of equations (2.2). Let T_N be given by:

$$T_N = \left(\frac{1}{n} \sum_{j=1}^n \frac{v(\delta_j)(B_N + \tau_j I_N)}{1 + c_N \psi(\delta_j)} \right)^{-1}.$$

Let $\hat{\chi}_N = \frac{1}{N} \text{Tr} B_N T_N$ and $\hat{\gamma}_N = \frac{1}{N} \text{Tr} T_N$. Then,

$$\delta_j = \hat{\chi}_N + \tau_j \hat{\gamma}_N, \quad j = 1, \dots, n.$$

Noticing that $\hat{\chi}_N$ and $\hat{\gamma}_N$ satisfy:

$$\hat{\chi}_N = \frac{1}{N} \text{Tr} B_N \left(\frac{1}{n} \sum_{j=1}^n \frac{v(\hat{\chi}_N + \tau_j \hat{\gamma}_N)(B_N + \tau_j I_N)}{1 + c_N \psi(\hat{\chi}_N + \tau_j \hat{\gamma}_N)} \right)^{-1} \quad (2.3)$$

$$\hat{\gamma}_N = \frac{1}{N} \text{Tr} \left(\frac{1}{n} \sum_{j=1}^n \frac{v(\hat{\chi}_N + \tau_j \hat{\gamma}_N)(B_N + \tau_j I_N)}{1 + c_N \psi(\hat{\chi}_N + \tau_j \hat{\gamma}_N)} \right)^{-1}, \quad (2.4)$$

it is not difficult to see that solving the system of the n equations in (2.2) can be reduced to determining the solutions of a two equations system, whose solutions are $\hat{\chi}_N$ and $\hat{\gamma}_N$. The control of δ_j in Lemma 4.6 allow us to ensure that $\hat{\chi}_N$ and $\hat{\gamma}_N$ are uniformly bounded for enough large n a.s. Hence, there exists a subsequence over which $\hat{\gamma}_N$ and $\hat{\chi}_N$ converge to γ_∞ and χ_∞ . Taking the limits of both sides of (2.3) and (2.4), we obtain

$$\chi_\infty = \int \frac{y}{\int_0^{+\infty} \frac{v(\chi_\infty + t\gamma_\infty)(y+t)}{1+c\psi(\chi_\infty+t\gamma_\infty)} \nu(dt)} F^B(dy) \quad (2.5)$$

$$\gamma_\infty = \int \frac{1}{\int_0^{+\infty} \frac{v(\chi_\infty + t\gamma_\infty)(y+t)}{1+c\psi(\chi_\infty+t\gamma_\infty)} \nu(dt)} F^B(dy). \quad (2.6)$$

Such limits are unique since the solutions of the systems of equations (2.5) and (2.6) are unique in case they exist. The existence and unicity of the solutions of (2.5) and (2.6) essentially relies on showing that the following function

$$h : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$$

$$(x_1, x_2) \mapsto (h_1(x_1, x_2), h_2(x_1, x_2)) \triangleq \left(\int \frac{y}{\int_0^{+\infty} \frac{v(x_1+tx_2)(y+t)}{1+c\psi(x_1+tx_2)} \nu(dt)} F^B(dy), \int \frac{1}{\int_0^{+\infty} \frac{v(x_1+tx_2)(y+t)}{1+c\psi(x_1+tx_2)} \nu(dt)} F^B(dy) \right)$$

is a standard interference function [20], i.e it satisfies the three conditions of positivity, monotonicity and scalability that have been used in the proof of Theorem 2.1. \square

3. NUMERICAL ANALYSIS

In order to assess the accuracy of our results, we represent in Fig. 1, the empirical estimate of the mean squared error (MSE) between the robust scatter estimate and \hat{S}_N with respect to N

$$\text{MSE} = \mathbb{E} \left\| \hat{S}_N - \hat{C}_N \right\|^2$$

when $n = 3N$ and $B_N = A_N A_N^*$ with A_N is $N \times \frac{N}{2}$ having independent standard Gaussian entries with zero mean and variance $\frac{1}{K}$. We set $u(t) = \frac{1+\alpha}{t+\alpha}$, and $\alpha = 0.5$. We note that the MSE decreases with N , thereby supporting the convergence of \hat{C}_N to \hat{S}_N .

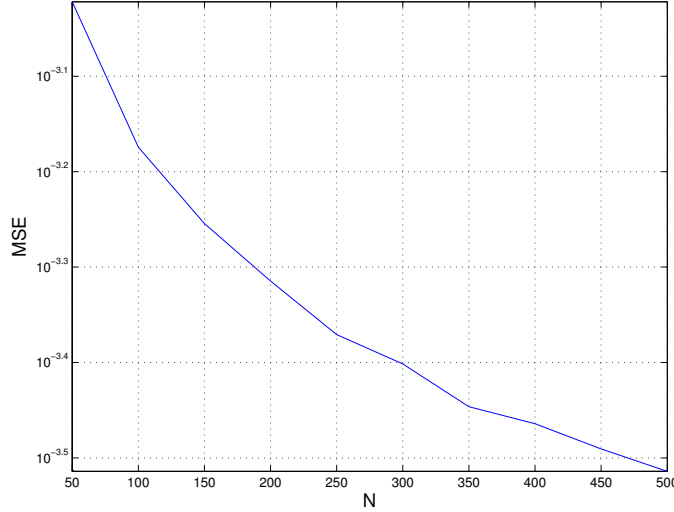


FIGURE 1. MSE with respect to N .

4. PROOFS

4.1. Heuristic Analysis. The study of the asymptotic behaviour of robust scatter matrices requires careful attention. The difficulty essentially lies in the rank-1 matrices present in the sum of (2.1) being dependent through \hat{C}_N . At first sight, this observation might make us

think that the asymptotic analysis of \hat{C}_N is out of the framework of the standard random matrix theory. However, a careful investigation of the expression of \hat{C}_N can lead us to replace \hat{C}_N by a random object, whose analysis using the theory of random matrices is quite standard.

Hereafter, we develop some heuristics that will lead to determine the asymptotic random equivalent of \hat{C}_N . We believe that beyond their interest to the considered scenario, these heuristics can facilitate the understanding of the asymptotic behaviour of robust estimation techniques in the regime where the number of observations is of the same order of the size of the population covariance matrix.

Building on the ideas of [8], we will first start by deriving a new rewriting of \hat{C}_N that will also be extensively used in section 4.2 devoted to the exposition of the rigorous proofs. Let $\hat{C}_{(i)}$ be the matrix \hat{C}_N where we remove $\frac{1}{n}u(\frac{1}{N}y_i^*\hat{C}_N^{-1}y_i)y_iy_i^*$, i.e.,

$$\hat{C}_{(i)} = \hat{C}_N - \frac{1}{n}u\left(\frac{1}{N}y_i^*\hat{C}_N^{-1}y_i\right)y_iy_i^*.$$

Applying the identity:

$$(A - tzz^*)^{-1}z = \frac{A^{-1}z}{1 - tzz^*A^{-1}z}$$

for any invertible A , vector z and scalar t such that $A - tzz^*$ is invertible, we obtain:

$$\begin{aligned} \frac{1}{N}y_i^*\hat{C}_{(i)}^{-1}y_i &= \frac{\frac{1}{N}y_i^*\hat{C}_N y_i}{1 - \frac{1}{n}y_i^*\hat{C}_N^{-1}y_i u(\frac{1}{N}y_i^*\hat{C}_N^{-1}y_i)} \\ &= \frac{\frac{1}{N}y_i^*\hat{C}_N y_i}{1 - c_N\phi\left(\frac{1}{N}y_i^*\hat{C}_N^{-1}y_i\right)} \\ &= g_N\left(\frac{1}{N}y_i^*\hat{C}_N^{-1}y_i\right), \end{aligned}$$

where $g_N : [0, \infty) \rightarrow [0, \infty)$, $x \mapsto \frac{x}{1 - c_N\phi(x)}$. As ϕ is increasing and $\phi_\infty < c_N^{-1}$, function g_N is positive increasing and maps $[0, \infty)$ to $[0, \infty)$. It is therefore invertible with inverse denoted by g_N^{-1} . We have thus:

$$\frac{1}{N}y_i^*\hat{C}_N^{-1}y_i = g_N^{-1}\left(\frac{1}{N}y_i^*\hat{C}_{(i)}^{-1}y_i\right).$$

We can therefore express \hat{C}_N as:

$$\begin{aligned} \hat{C}_N &= \frac{1}{n}\sum_{j=1}^n \left(u \circ g_N^{-1}\right)\left(\frac{1}{N}y_j^*\hat{C}_{(j)}^{-1}y_j\right)y_jy_j^* \\ &= \frac{1}{n}\sum_{j=1}^n v\left(\frac{1}{N}y_j^*\hat{C}_{(j)}^{-1}y_j\right)y_jy_j^* \end{aligned}$$

with $v = u \circ g_N^{-1}$ positive and non-increasing.

This new rewriting of \hat{C}_N is of fundamental importance. It has two major advantages. First, it reveals that \hat{C}_N is uniquely determined by $q_j = \frac{1}{N}y_j^*\hat{C}_{(j)}^{-1}y_j$, $j = 1, \dots, n$. This can be seen by noticing that a solution \hat{C}_N to (2.1) exists and is unique if and only if the

following system of equations in x_1, \dots, x_n :

$$x_j = y_j^* \left(\frac{1}{n} \sum_{i=1, i \neq j}^n v(x_i) y_i y_i^* \right)^{-1} y_j$$

admits a unique positive solution q_1, \dots, q_n . The estimation of the $N \times N$ robust scatter matrix is then reduced to determining the solutions of a n system of equations. The second advantage of this new rewriting is that it can provide, based on some heuristics, interesting insights about the asymptotic behaviour of \hat{C}_N . In effect, it is not difficult to understand that y_i is weakly dependent on $\hat{C}_{(i)}$, since $\hat{C}_{(i)}$ depends on y_i only through the terms $\frac{1}{N} y_j^* \hat{C}_N^{-1} y_j, j \neq i$. Standard results from random matrix theory will thus lead to $q_i = \frac{1}{N} y_i^* \hat{C}_{(i)}^{-1} y_i \sim \frac{1}{N} \text{Tr}(B_N + \tau_i I_N) \hat{C}_{(i)}^{-1}$, which tends to imply that q_i scales with τ_i . Assume that $q_i, i = 1, \dots, n$ can be approximated by δ_i where δ_i does not depend on the random vector w_i . Then, because of rank-1 perturbation arguments leading to replace $\hat{C}_{(i)}^{-1}$ with \hat{C}_N^{-1} , we have:

$$\frac{q_i}{\frac{1}{N} \text{Tr}(B_N + \tau_i I_N) \hat{C}_{(i)}^{-1}} \sim \frac{\delta_i}{\frac{1}{N} \text{Tr}(B_N + \tau_i I_N) \hat{C}_N^{-1}}.$$

On the other hand, from the asymptotic equivalence between q_i and δ_i , we expect \hat{C}_N to be asymptotically equivalent to $\frac{1}{n} \sum_{i=1}^n v(\delta_i) y_i y_i^*$. As we will see later, without inducing a major error, one can assume that y_i are Gaussian. The asymptotic behaviour of $\frac{1}{n} \sum_{i=1}^n v(\delta_i) y_i y_i^*$ can be thus studied using results from [19]. If Theorem 1 in [19] is applicable, then δ_i should satisfy:

$$\begin{aligned} 1 &\sim \frac{1}{N} \text{Tr} \frac{(B_N + \tau_i I_N)}{\delta_i} \hat{C}_N^{-1} \\ &\sim \frac{1}{N} \text{Tr} \frac{(B_N + \tau_i I_N)}{\delta_i} \left(\frac{1}{n} \sum_{j=1}^n \frac{v(\delta_j) (B_N + \tau_j I_N)}{1 + e_j} \right)^{-1}, \end{aligned} \quad (4.1)$$

where e_1, \dots, e_n are the fixed point solutions to the following system of equations:

$$e_i = \frac{v(\delta_i)}{n} \text{Tr}(B_N + \tau_i I_N) \left(\frac{1}{n} \sum_{j=1}^n \frac{v(\delta_j) (B_N + \tau_j I_N)}{1 + e_j} \right)^{-1}.$$

Multiplying both sides of (4.1), we thus get:

$$e_i \sim c_N \delta_i v(\delta_i) = c_N \psi(\delta_i).$$

Plugging the above equations into (4.1), we obtain that $\delta_1, \dots, \delta_n$ are solutions to the following system of equations:

$$\delta_i = \frac{1}{N} \text{Tr}(B_N + \tau_i I_N) \left(\frac{1}{n} \sum_{j=1}^n \frac{v(\delta_j) (B_N + \tau_j I_N)}{1 + c_N \psi(\delta_j)} \right), \quad i = 1, \dots, n.$$

4.2. Rigorous Proofs. The main differences of our work with respect to the one in [8] lies in the considered data model. While [8] assumes purely noise observations drawn from elliptical distributions, we consider in the present work, sequence of time observations that are given by the sum of a heavy-tailed noise and a Gaussian distributed vector modeling the "signal" part of the observations. In practice, the estimation of the covariance matrix

of the available observations can help infer precious information on the signal of interest. From a theoretical standpoint, if the useful data live in a low-dimensional space, the same approach considered in [8] can be pursued with only minor changes. Although less popular, high rank data models, occurring when K scales with N , are more attractive for several applications of array processing concerning distributed source localization [17]. They are also more difficult to handle, since the use of the approach of [8] poses many technical difficulties, when B_N is allowed to be of high rank. This can be easily seen by noticing that our heuristic computations involve solving a system of n equations while those of [8] requires only solving the fixed point of a single equation. One can easily convince oneself that in the context of interest, it is much more difficult to get insights into the behaviour of the n solutions of the underlying system. Before delving into the core of the proof, we need first to introduce in the sequel some preliminary results that will help adapt the techniques of [8] to our particular context.

4.2.1. Preliminary Results.

Function u and Related Functions. The robust-scatter estimator is parametrized by function u , which significantly impacts its performance. This intuition is further confirmed by theoretical analysis, showing that a number of sequence of functions in relation with u naturally arise. This section aims at presenting the list of these functions along with some of their most important properties. We first summarize in the following table some of the results that has been established in [8].

TABLE 1. Properties of Functions

Functions	Properties
$u(x)$	Non-increasing Positive Continuous
$\phi(x) \triangleq xu(x)$	Increasing Positive Continuous Bounded with $\phi_\infty < c_+^{-1}$.

Sequence of Functions	Properties
$g_N(x) \triangleq \frac{x}{1 - c_N \phi(x)}$	Increasing Positive Continuous Unbounded
$v_N(x) \triangleq u \circ g_N^{-1}(x)$	Non-Increasing Positive Continuous
$\psi_N(x) \triangleq xv(x)$	Increasing Positive Continuous Bounded with $\psi_\infty = \frac{\phi_\infty}{1 - c_N \phi_\infty}$

In addition to the aforementioned properties, we need to prove the following results, which will be used in our proofs.

Lemma 4.1. *Let $u(\cdot)$ and $\phi(\cdot)$ be two functions satisfying assumption 2. Then, we have, for all $x, y \geq 0$,*

$$\frac{\phi(x) - \phi(y)}{x - y} \leq u(x) \leq u(0).$$

In other words, $\phi(\cdot)$ is Lipschitz with Lipschitz constant $u(0)$.

Proof. We have:

$$\begin{aligned} \frac{\phi(x) - \phi(y)}{x - y} &= \frac{xu(x) - yu(y)}{x - y} \\ &= \frac{xu(x) - yu(x) + yu(x) - yu(y)}{x - y}. \end{aligned}$$

Since $u(\cdot)$ is non-increasing:

$$y \frac{u(x) - u(y)}{x - y} \leq 0.$$

Therefore,

$$\frac{\phi(x) - \phi(y)}{x - y} \leq \frac{xu(x) - yu(x)}{x - y} = u(x) \leq u(0).$$

□

Lemma 4.2. *Let u and ϕ be two functions satisfying assumption 2. Then, we have, for all $x \geq 0$:*

$$\phi(x) \leq u(0)x.$$

Moreover, for all $x \in [0, m]$,

$$u(m)x \leq \phi(x) \leq u(0)x.$$

Proof. The first statement follows from the previous lemma by setting $y = 0$. To prove the second, notice that when $x \leq m$, $u(x) \geq u(m)$, thereby showing that $\phi(x) \geq xu(m)$ whenever $x \in [0, m]$. □

Remark 4.1. *As it has been proven in [8], functions $x \mapsto \psi(x)$ and $x \mapsto v(x)$ share respectively the same properties as $x \mapsto \phi(x)$ and $x \mapsto u(x)$. As a consequence, we can prove that $x \mapsto \psi(x)$ is Lipschitz with constant $v(0) = u(0)$. The constant Lipschitz being independent on n , we conclude that (ψ_N) form an equicontinuous family of functions and as such converge uniformly on $[0, \infty)$. Moreover,*

$$\psi(x) \leq v(0)x$$

and $\psi(x) \geq v(m)x$ whenever $x \in [0, m]$.

Lemma 4.3. *Let $g_N(\cdot) : x \mapsto \frac{x}{1 - c_N \phi(x)}$. Denote by $g_N^{-1}(\cdot)$ the inverse function corresponding to g_N . Then, for all $y \geq z \geq 0$, we have:*

$$g_N^{-1}(y) - g_N^{-1}(z) \leq (y - z)(1 - c_N \phi(g_N^{-1}(y)))$$

In particular, g_N^{-1} is Lipschitz on $[0, \infty)$ with constant Lipschitz 1. Besides, functions $\left(x \mapsto \frac{\psi_N(x)}{1 + c_N \psi_N(x)}\right)_{N=1}^{\infty}$ are Lipschitz and converge uniformly on $[0, \infty)$.

Proof. Let $y \geq z \geq 0$. From the relation $g_N^{-1}(y) = y - c_N \phi(g_N^{-1}(y))$, we have:

$$\begin{aligned} g_N^{-1}(y) - g_N^{-1}(z) &= y - z + c_N z \phi(g_N^{-1}(z)) - c_N y \phi(g_N^{-1}(y)) \\ &= y - z + c_N z \left(\phi(g_N^{-1}(z)) - \phi(g_N^{-1}(y)) \right) + c_N (z - y) \phi(g_N^{-1}(y)). \end{aligned}$$

Since g_N^{-1} is increasing, $\phi(g_N^{-1}(z)) - \phi(g_N^{-1}(y)) \leq 0$. Hence,

$$g_N^{-1}(y) - g_N^{-1}(z) \leq (y - z)(1 - c_N \phi(g_N^{-1}(y))) \leq (y - z).$$

Finally, after simple calculations, we can prove that:

$$\frac{\psi_N(x)}{1 + c_N \psi_N(x)} = \phi \circ g_N^{-1}(x).$$

Therefore, $x \mapsto \frac{\psi_N(x)}{1 + c_N \psi_N(x)}$ is Lipschitz with constant lipschitz equal to $u(0)$. This constant being independent on n , the sequence of functions $\frac{\psi_N(x)}{1 + c_N \psi_N(x)}$ converge uniformly on $[0, \infty)$. \square

Useful results. As previously stated, the difficulty of studying the robust-scatter estimator lies in the control of the asymptotic behaviour of q_i and δ_i . The proof of Theorem 2.1 and Theorem 2.2 will require us to show that q_i and δ_i scale with τ_i and to control quadratic forms involving matrix $\frac{1}{n} \sum_{i=1}^n f(\tau_i) y_i y_i^*$ where f is a certain functional. To this end, we develop in this section two key results that will underlie the proof of the main theorems.

Proposition 4.4. *Let (B_N) be a sequence of $N \times N$ hermitian positive matrices satisfying Assumption 4-iii). In addition, let $\tau_i, i = 1, \dots, n$ be positive random variables satisfying assumption 4-ii). Consider (f_N) a sequence of piece-wise continuous positive bounded functions defined on $[0, \infty)$ that has at least one subsequence converging uniformly. We assume that functions $t \mapsto f_N(t)$ satisfy the following additional properties:*

- *Function $t \mapsto f_N(t)$ grows at most linearly, i.e there exists $\alpha, \beta > 0$ such that:*

$$\begin{aligned} \sup_N f_N(t) &\leq \alpha \quad \forall t \geq 0, \\ \sup_N f_N(t) &\leq \beta t \quad \forall t \geq 0. \end{aligned}$$

- $\int f_N(t) \nu(dt) = 1$
- $\liminf_N \inf_{t \in [m, +\infty)} f_N(t) > 0$.

Then the following equation in x :

$$\int \frac{F^{B_N}(dy)}{\int \frac{y+t}{t+x} f_N(t) \nu(dt)} = 1 \quad (4.2)$$

admits a unique positive solution which we denote by η_N . Then, there exists a sequence (r_N^-) with $\liminf r_N^- > 0$ such that:

$$r_N^- \frac{1}{N} \text{Tr } B_N \leq \eta_N \leq \frac{1}{N} \text{Tr } B_N.$$

Moreover, we have:

$$c_N - |\epsilon_{n,j}| \leq \frac{1}{N} \text{Tr } \frac{B_N + \tau_j I}{\tau_j + \eta_N} \left(\frac{1}{N} \sum_{i=1}^n f_N(\tau_i) \frac{B_N + \tau_i I}{\tau_i + \eta_N} \right)^{-1} \leq c_N + |\epsilon_{n,j}|, \quad (4.3)$$

where $\max_{1 \leq j \leq n} |\epsilon_{n,j}|$ converges almost surely to zero.

Proof. We start by showing that (4.2) admits a unique solution η_N . It is clear that function $h_N : x \mapsto \int \frac{F^{B_N}(dy)}{\int \frac{y+t}{t+x} f_N(t) \nu(dt)}$ is increasing and continuous on $(0, +\infty)$ with the limit at $x \rightarrow 0^+$ less than 1, while the limit when $x \rightarrow +\infty$ is $+\infty$. Therefore, there exists a unique η_N that satisfies (4.2). It is easy to check that η_N is less than the maximum eigenvalue of B_N . Therefore, we can restrict the domain of h_N to the set $[0, \|B_N\|]$. Since $y \mapsto \frac{1}{\int \frac{y+t}{t+x} f_N(t) \nu(dt)}$ is convex, applying the Jensen inequality, we obtain:

$$\int \frac{F^{B_N}(dy)}{\int \frac{y+t}{t+x} f_N(t) \nu(dt)} \geq \frac{1}{\int \frac{\frac{1}{N} \text{Tr } B_N + t}{t+x} f_N(t) \nu(dt)}.$$

Setting $x = \eta_N$, the above inequality becomes:

$$1 \geq \frac{1}{\int \frac{\frac{1}{N} \text{Tr } B_N + t}{t+\eta_N} f_N(t) \nu(dt)}. \quad (4.4)$$

Therefore, $\eta_N \leq \frac{1}{N} \text{Tr } B_N$ because otherwise, (4.4) would not hold.

The proof of the lower-bound inequality is more delicate. Let m be as in Assumption 3-ii) and denote by $h_{m,N}$ the following map:

$$h_{m,N} : [0, \|B_N\|] \rightarrow \mathbb{R}^+, x \mapsto \int \frac{1}{y \int_m^\infty \frac{f_N(t)}{t+x} \nu(dt) + \int_0^\infty \frac{t f_N(t)}{t+x} \nu(dt)} F^{B_N}(dy).$$

Functions h_N and $h_{m,N}$ are both increasing, while $h_{m,N}(x) \geq h_N(x) \quad \forall x \in [0, \|B_N\|]$. Furthermore, we can easily check that:

$$\lim_{x \rightarrow 0^+} h_{m,N}(x) < 1 \text{ and } \lim_{x \rightarrow +\infty} h_{m,N}(x) = +\infty.$$

Therefore, there exists $\eta_{N,m}$ solution in x to the equation $h_{m,N}(x) = 1$. Moreover, we have $h_N(\eta_{N,m}) \leq 1$, and thus $\eta_N \geq \eta_{N,m}$. On the other hand, $h_{m,N}$ is differentiable with derivative $h'_{m,N}(x)$ given by:

$$h'_{m,N}(x) = \int \frac{y \int_m^{+\infty} \frac{f_N(t)}{(t+x)^2} \nu(dt) + \int_0^{+\infty} \frac{t f_N(t)}{(t+x)^2} \nu(dt)}{\left(y \int_m^{+\infty} \frac{f_N(t)}{t+x} \nu(dt) + \int_0^{+\infty} \frac{t f_N(t)}{t+x} \nu(dt) \right)^2} F^{B_N}(dy).$$

Let $a_{m,N} = \int_m^{+\infty} \frac{f_N(t)}{t} \nu(dt)$. Hence, if $0 \leq x \leq \|B_N\|$,

$$\begin{aligned} h'_{m,N}(x) &\leq \int \frac{y \int_m^{+\infty} \frac{f_N(t)}{(t+x)^2} \nu(dt) + \int_0^{+\infty} \frac{t f_N(t)}{(t+x)^2} \nu(dt)}{\left(\int_m^{+\infty} \frac{t f_N(t)}{t+x} \nu(dt) \right)^2} F^{B_N}(dy) \\ &\leq \int \frac{y \int_m^{+\infty} \frac{f_N(t)}{mt} \nu(dt) + \int_0^{+\infty} \frac{f_N(t)}{t} \nu(dt)}{\left(\frac{m^2}{\|B_N\|+m} a_{m,N} \right)^2} F^{B_N}(dy) \\ &\leq \frac{\beta \left(\frac{\|B_N\|}{m} + 1 \right)^3}{m^2 a_{m,N}^2}. \end{aligned}$$

The mean value theorem implies that:

$$\frac{1 - h_{m,N}(0)}{\eta_{N,m}} \leq \frac{\beta \left(\frac{\|B_N\|}{m} + 1 \right)^3}{m^2 a_{m,N}^2}, \quad (4.5)$$

where

$$h_{m,N}(0) = \int \frac{1}{y \int_m^{+\infty} \frac{f_N(t)}{t} + 1} F^{B_N}(dy) = \frac{1}{N} \text{Tr} (a_{m,N} B_N + I_N)^{-1}.$$

As a consequence,

$$\begin{aligned} 1 - h_{m,N}(0) &= \frac{1}{N} \text{Tr} I_N - \frac{1}{N} \text{Tr} (a_{m,N} B_N + I_N)^{-1} \\ &= \frac{a_{m,N}}{N} \text{Tr} B_N (a_{m,N} B_N + I_N)^{-1} \geq \frac{a_{m,N} \frac{1}{N} \text{Tr} B_N}{a_{m,N} \|B_N\| + 1}. \end{aligned} \quad (4.6)$$

Combining (4.5) and (4.6), we therefore get:

$$\eta_N \geq \frac{m^2 a_{m,N}^3 \frac{1}{N} \text{Tr} B_N}{(a_{m,N} \|B_N\| + 1) \left(1 + \frac{\|B_N\|}{m}\right)^3 \beta}.$$

Note that it is easy to prove that $r_N \triangleq \frac{m^2 a_{m,N}^3}{(a_{m,N} \|B_N\| + 1) \left(1 + \frac{\|B_N\|}{m}\right)^3 \beta} \leq 1$, since $ma_{m,N} \leq 1$ and $\frac{a_{m,N}}{\beta} < 1$. Moreover, $\liminf r_N > 0$ as $r_N \geq \frac{m^2 a_{m,N}^3}{\beta(\beta \|B_N\| + 1)^4}$ and $\liminf a_{m,N} \geq \liminf_N \inf_{x \in [m, \infty)} f_N(x) \int_m^{+\infty} \frac{1}{t} \nu(dt) > 0$.

We will now proceed proving the inequalities in (4.3). Let $\lambda_1^N \leq \dots \leq \lambda_N^N$ be the eigenvalues of B_N . We have:

$$\begin{aligned} \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I}{\tau_j + \eta_N} \left(\frac{1}{N} \sum_{i=1}^n f_N(\tau_i) \frac{B_N + \tau_i I}{\tau_i + \eta_N} \right)^{-1} &= \frac{1}{N} \sum_{k=1}^N \frac{\lambda_k^N + \tau_j}{(\tau_j + \eta_N) \frac{1}{N} \sum_{i=1}^n f_N(\tau_i) \frac{\lambda_k^N + \tau_i}{\tau_i + \eta_N}} \\ &= \frac{c_N}{\tau_j + \eta_N} \int \frac{y F^{B_N}(dy)}{\frac{1}{n} \sum_{i=1}^n f_N(\tau_i) \frac{y + \tau_i}{\tau_i + \eta_N}} + \frac{c_N \tau_j}{(\tau_j + \eta_N)} \int \frac{F^{B_N}(dy)}{\frac{1}{n} \sum_{i=1}^n f_N(\tau_i) \frac{y + \tau_i}{\tau_i + \eta_N}} \\ &= \frac{c_N}{\tau_j + \eta_N} \int \frac{y dF^{B_N}}{\int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)} + \frac{c_N \tau_j}{\tau_j + \eta_N} \int \frac{dF^{B_N}}{\int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)} + \frac{c_N}{\tau_j + \eta_N} \epsilon_{n,1} + \frac{c_N \tau_j}{\tau_j + \eta_N} \epsilon_{n,2}, \end{aligned} \quad (4.7)$$

where

$$\epsilon_{n,1} = \int \frac{y \left(\int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt) - \frac{1}{n} \sum_{i=1}^n f_N(\tau_i) \frac{y + \tau_i}{\tau_i + \eta_N} \right)}{\frac{1}{n} \sum_{i=1}^n f_N(\tau_i) \frac{y + \tau_i}{\tau_i + \eta_N} \int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)} F^{B_N}(dy) \quad (4.8)$$

$$\epsilon_{n,2} = \int \frac{\int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt) - \frac{1}{n} \sum_{i=1}^n f_N(\tau_i) \frac{y + \tau_i}{\tau_i + \eta_N}}{\frac{1}{n} \sum_{i=1}^n f_N(\tau_i) \frac{y + \tau_i}{\tau_i + \eta_N} \int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)} F^{B_N}(dy). \quad (4.9)$$

As η_N is the unique solution of (4.2), $\int \frac{y F^{B_N}(dy)}{\int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)}$ can be further simplified as:

$$\begin{aligned} \int \frac{y F^{B_N}(dy)}{\int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)} &= \int \frac{y \int \frac{f_N(t)}{t+\eta_N} \nu(dt) + \int \frac{t f_N(t)}{t+\eta_N} \nu(dt) - \int \frac{t f_N(t)}{t+\eta_N} \nu(dt)}{\int \frac{f_N(t)}{t+\eta_N} \nu(dt) \int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)} F^{B_N}(dy) \\ &\stackrel{(a)}{=} \frac{\int f_N(t) \nu(dt)}{\int \frac{f_N(t)}{t+\eta_N} \nu(dt)} - \frac{\int \frac{t f_N(t)}{t+\eta_N} \nu(dt)}{\int \frac{f_N(t)}{t+\eta_N} \nu(dt)} \\ &= \eta_N, \end{aligned} \quad (4.10)$$

where (a) follows due to the fact that $\int \frac{1}{\int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)} F^{B_N}(dy) = 1$. Substituting (4.10) into (4.7), we get:

$$\frac{1}{N} \text{Tr} \frac{B_N + \tau_j I}{\tau_j + \eta_N} \left(\frac{1}{N} \sum_{i=1}^n f(\tau_i) \frac{B_N + \tau_i I}{\tau_i + \eta_N} \right)^{-1} = c_N + \frac{c_N}{\tau_j + \eta_N} \epsilon_{n,1} + \frac{c_N \tau_j}{\tau_j + \eta_N} \epsilon_{n,2}.$$

Therefore, the result immediately follows once we prove that $\max_{1 \leq j \leq n} \frac{1}{\tau_j + \eta_N} |\epsilon_{n,1}|$ and $|\epsilon_{n,2}|$ converge almost surely to zero. We will only control $\frac{1}{\tau_j + \eta_N} \epsilon_{n,1}$. The control of $\epsilon_{n,2}$ can be obtained using the same arguments. We have:

$$\begin{aligned} \frac{|\epsilon_{n,1}|}{\tau_j + \eta_N} &\leq \sup_y \left| \int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt) - \frac{1}{n} \sum_{i=1}^n f_N(\tau_i) \frac{y+\tau_i}{\tau_i + \eta_N} \right| \\ &\quad \times \frac{1}{\eta_N} \int \frac{y F^{B_N}(dy)}{\left(\frac{1}{n} \sum_{i=1}^n f_N(\tau_i) \frac{y+\tau_i}{\tau_i + \eta_N} \right) \int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)} \\ &\leq \left(\lambda_N^N \left| \int \frac{f_N(t)}{t+\eta_N} \nu(dt) - \frac{1}{n} \sum_{i=1}^n \frac{f_N(\tau_i)}{\tau_i + \eta_N} \right| + \left| \int \frac{t f_N(t)}{t+\eta_N} \nu(dt) - \frac{1}{n} \sum_{i=1}^n \frac{f_N(\tau_i) \tau_i}{\tau_i + \eta_N} \right| \right) \\ &\quad \times \frac{1}{\eta_N} \int \frac{y F^{B_N}(dy)}{\left(\frac{1}{n} \sum_{i=1}^n \frac{f_N(\tau_i)(y+\tau_i)}{\tau_i + \eta_N} \right) \int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)}. \end{aligned}$$

Since $\frac{f_N(t)}{t+\eta_N} \leq \frac{\beta t}{t+\eta_N} \leq \beta$ and $\frac{t f_N(t)}{t+\eta_N} \leq \frac{t \alpha}{t+\eta_N} \leq \alpha$, sequences $\int \frac{f_N(t)}{t+\eta_N} \nu(dt) - \int \frac{f(t)}{t+\eta_N} \nu(dt)$ and $\int \frac{t f_N(t)}{t+\eta_N} \nu(dt) - \int \frac{t f(t)}{t+\eta_N} \nu(dt)$ are bounded. One can extract a subsequence (n) such that: $\int \frac{f_N(t)}{t+\eta_N} \nu(dt) - \int \frac{f(t)}{t+\eta_N} \nu(dt)$ and $\int \frac{t f_N(t)}{t+\eta_N} \nu(dt) - \int \frac{t f(t)}{t+\eta_N} \nu(dt)$ converge. Over this subsequence, $t \mapsto \frac{t f_N(t)}{t+\eta_N}$ converge uniformly to $t \mapsto \frac{t f^*(t)}{t+\eta^*}$ and as such:

$$\int \frac{t f_N(t)}{t+\eta_N} \nu(dt) - \int \frac{t f^*(t)}{t+\eta^*} \nu(dt) \xrightarrow{\text{a.s.}} 0.$$

Moreover, since $\liminf \eta_N \geq \liminf r_N^{-1} \frac{1}{N} \text{Tr} B_N > 0$, $\eta^* \neq 0$. The sequence of functions $t \mapsto \frac{f_N(t)}{t+\eta_N}$ converge also uniformly to $t \mapsto \frac{f^*(t)}{t+\eta^*}$, thereby yielding:

$$\int \frac{f_N(t)}{t+\eta_N} \nu(dt) - \int \frac{f^*(t)}{t+\eta^*} \nu(dt) \xrightarrow{\text{a.s.}} 0.$$

It remains thus to check that $\frac{1}{\eta_N} \int \frac{y dF^{B_N}}{\left(\frac{1}{n} \sum_{i=1}^n \frac{f_N(\tau_i)(y+\tau_i)}{\tau_i + \eta_N} \right) \int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)}$ is almost surely bounded.

For that, first note that since $\eta_N = \int \frac{y dF^{B_N}}{\int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)}$, we have:

$$\frac{1}{\eta_N} \int \frac{y}{\frac{1}{n} \sum_{i=1}^n f_N(\tau_i) \frac{y+\tau_i}{\tau_i + \eta_N} \int \frac{y+t}{t+\eta_N} f_N(t) \nu(dt)} dF^{B_N} \leq \frac{1}{\frac{1}{n} \sum_{\tau_i \geq m}^n \frac{f_N(\tau_i) m}{m+\eta_N}}$$

As $\eta_N \leq \lambda_N^N$ and $\liminf_N \inf_{t \in [m, +\infty)} f_N(t) > 0$, $\frac{1}{n} \sum_{\tau_i \geq m}^n \frac{f_N(\tau_i) m}{m+\eta_N}$ is almost surely bounded away from zero, thereby implying the desired result. The control of $\epsilon_{n,2}$ could be done using the same arguments. \square

The second ingredient that will be of extensive use in the proof of the theorem is provided by the following key-lemma.

Lemma 4.5. *Let Assumption 1-4 hold true. Let (f_N) be a sequence functions satisfying the conditions of proposition 4.4. Denote by η_N the unique solution in x to the following equation:*

$$\int_0^\infty \frac{F^{B_N}}{\frac{y+t}{t+x} f_N(t) \nu(dt)} = 1.$$

Consider e_1, \dots, e_n the unique solutions to the following system of equations:

$$e_k = \frac{f_N(\tau_k)}{n} \text{Tr} \frac{B_N + \tau_k I_N}{\tau_k + \eta_N} \left(\frac{1}{n} \sum_{i=1}^n \frac{f_N(\tau_i)(B_N + \tau_i I_N)}{(\tau_i + \eta_N)(1 + e_i)} \right)^{-1}. \quad (4.11)$$

Then, the following statements hold true:

$$i) \max_{1 \leq j \leq n} \left| \frac{f_N(\tau_j)}{N(\tau_j + \eta_N)} y_j^* \left(\frac{1}{n} \sum_{i \neq j} \frac{f_N(\tau_i)}{\tau_i + \eta_N} y_i y_i^* \right)^{-1} y_j - c_N^{-1} e_j \right| \xrightarrow{a.s.} 0.$$

$$ii) \text{ If } T_N = \left(\frac{1}{n} \sum_{i=1}^n \frac{f_N(\tau_i)(B_N + \tau_i I_N)}{(\tau_i + \eta_N)(1 + e_i)} \right)^{-1}. \text{ Then, we have:}$$

$$\max_{1 \leq j \leq n} \left| \frac{1}{N(\tau_j + \eta_N)} y_j^* \left(\frac{1}{n} \sum_{i \neq j} \frac{f_N(\tau_i)}{\tau_i + \eta_N} y_i y_i^* \right)^{-1} y_j - \frac{1}{N} \text{Tr} \frac{(B_N + \tau_j I_N)}{\tau_j + \eta_N} T_N \right| \xrightarrow{a.s.} 0. \quad (4.12)$$

iii) If $\|f_N\|_\infty < \frac{1}{c_N}$, then, there exists $\epsilon_n \downarrow 0$ such that for n large enough, a.s.

$$\max_{1 \leq k \leq n} e_k \leq \frac{c_N \|f_N\|_\infty}{1 - \|f_N\|_\infty c_N} + \epsilon_n.$$

Proof. The proof of the first two items is based on Lemma .15 and Lemma .16 in Appendix 4.2.2. For these Lemmas to be applicable, we need to check that $\liminf_N \min_{1 \leq j \leq n} \lambda_1 \left(\frac{1}{n} \sum_{i \neq j} \frac{f_N(\tau_i)}{\tau_i + \eta_N} y_i y_i^* \right) > 0$. To this end, first note that:

$$\min_{1 \leq j \leq n} \lambda_1 \left(\frac{1}{n} \sum_{i \neq j} \frac{f_N(\tau_i)}{\tau_i + \eta_N} y_i y_i^* \right) \geq \min_{1 \leq j \leq n} \lambda_1 \left(\frac{1}{n} \sum_{i \neq j, \tau_i \geq m} \frac{f_N(\tau_i)}{\tau_i + \eta_N} y_i y_i^* \right).$$

The right-hand side of the above equality is almost surely bounded above zero since $f_N(\tau_i) \frac{(B_N + \tau_i I_N)}{\tau_i + \eta_N} \succeq \inf_{t \in [m, \infty)} f_N(t) \frac{m}{m + \eta_N}$ and η_N is almost surely bounded by proposition 4.4. We conclude thus by resorting to Lemma .14 in Appendix 4.2.2.

In order to prove the last statement, let j_0 be the index of the maximum element in $\{e_1, \dots, e_n\}$. We therefore have:

$$\begin{aligned} e_{j_0} &\leq \|f_N\|_\infty (1 + e_{j_0}) \frac{1}{n} \text{Tr} \frac{(B_N + \tau_{j_0} I_N)}{\tau_{j_0} + \eta_N} \left(\frac{1}{n} \sum_{i=1}^n \frac{f_N(\tau_i)(B_N + \tau_i I_N)}{\tau_i + \eta_N} \right)^{-1} \\ &\stackrel{(a)}{\leq} \|f_N\|_\infty (1 + e_{j_0}) (c_N + \epsilon_n), \end{aligned}$$

where (a) follows from proposition 4.4. Besides, scalars e_1, \dots, e_n being the limits of almost surely bounded random quantities are bounded. Therefore,

$$e_{j_0} \leq \frac{\|f_N\|_{\infty} c_N}{1 - \|f_N\|_{\infty} c_N} + \epsilon'_n$$

where $\epsilon'_n \downarrow 0$. □

4.2.2. Proof of the Main Theorems. With the above preliminary results at hand, we are now in position to provide the proofs of Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1: Asymptotic Existence of the Robust Scatter Estimator.

Theorem 2.1 establishes the existence of the robust scatter estimate for large n and N . In particular, it implies that for each realization, there exists n_0 and N_0 large such that for all n and N greater than n_0 and N_0 , equation (2.1) admits a unique solution. Although we believe that a stronger result showing the existence of the robust scatter estimate for well-behaved set of samples can be established using probably the same kind of techniques as in [5], we have chosen in this paper to show Theorem 2.1 under the setting of the asymptotic regime. The reason is that the techniques used in that proof will be key to understanding some aspects of the asymptotic behaviour of the robust scatter estimate, thereby paving the way towards the proof of Theorem 2.2.

The proof of Theorem 2.1 follows the same lines as in [8]. Define $h = (h_1, \dots, h_n)$ with:

$$h_j : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$$

$$(x_1, \dots, x_n) \mapsto \frac{1}{N} y_j^* \left(\frac{1}{n} \sum_{\substack{i=1 \\ i \neq j}}^n v(x_i) y_i y_i^* \right)^{-1} y_j.$$

As it has been already mentioned, in order to prove that \hat{C}_N is uniquely defined for n large enough a.s., it suffices to show that the system of equations in x_1, \dots, x_n

$$x_j = h_j(x_1, \dots, x_n), j = 1, \dots, n$$

admits a unique solution q_1, \dots, q_n a.s. for n large enough. To this end, we will show that h is a standard interference function, i.e, it satisfies the following three conditions:

- a) Positivity: For each $q_1, \dots, q_n \geq 0$, and each i , $h_i(q_1, \dots, q_n) > 0$,
- b) Monotonicity: For each $q_1 \geq q'_1, \dots, q_n \geq q'_n$ and each i , $h_i(q_1, \dots, q_n) \geq h_i(q'_1, \dots, q'_n)$,
- c) Scalability: For all $\alpha > 1$, and $q_1, \dots, q_n \geq 0$, $\alpha h_i(q_1, \dots, q_n) > h_i(\alpha q_1, \dots, \alpha q_n)$.

Item a) can be easily shown by noticing that matrix $\frac{1}{n} \sum_{i=1, i \neq j}^n v(q_i) y_i y_i^*$ is invertible almost surely and is positive definite, while the monotonicity follows immediately from the fact that h_j is non-decreasing of each q_i . As for the scalability, we can assume without loss of generality that there exists $q_i > 0$ as the results holds trivially when $q_1 = \dots = q_n = 0$. With this assumption at hand, we rewrite $h_j(q_1, \dots, q_n)$ as:

$$h_j(q_1, \dots, q_n) = \frac{1}{N} y_j^* \left(\frac{1}{n} \sum_{\substack{i=1 \\ i \neq j}}^n \frac{\psi_N(q_i)}{q_i} y_i y_i^* \right)^{-1} y_j.$$

As ψ_N is increasing, $\psi_N(\alpha q) > \psi_N(q)$ for $\alpha > 1$ and $q > 0$. Hence,

$$h_j(\alpha q_1, \dots, \alpha q_n) = \frac{\alpha}{N} y_j^* \left(\frac{1}{n} \sum_{\substack{i=1 \\ i \neq j}}^n \frac{\psi_N(\alpha q_i)}{q_i} y_i y_i^* \right)^{-1} y_j.$$

If there exists at least $q_i > 0$, we therefore get:

$$h_j(\alpha q_1, \dots, \alpha q_n) < \frac{\alpha}{N} y_j^* \left(\frac{1}{n} \sum_{i=1, i \neq j}^n \frac{\psi_N(q_i)}{q_i} y_i y_i^* \right)^{-1} y_j = \alpha h_j(q_1, \dots, q_n).$$

We have thus established that h is a standard interference function. Referring to the results of [20], it remains to show that there exists vector (q_1, \dots, q_n) such that for all $i = 1, \dots, n$, $q_i > h_i(q_1, \dots, q_n)$ a.s. for n large enough, a statement which is known as the feasibility condition.

In order to establish the feasibility condition, let q_N^+ be chosen so that:

$$\int_0^{+\infty} \psi_N(q_N^+ t) \nu(dt) = \frac{1 + \kappa}{1 - c_N \phi_\infty}.$$

for some sufficiently small $0 \leq \kappa \leq 1$ satisfying:

$$1 + \kappa \leq \phi_\infty(1 - \nu\{0\}).$$

This is possible since,

$$\lim_{q \rightarrow +\infty} \int_0^{+\infty} \psi_N(qt) \nu(dt) = \psi_\infty(1 - \nu\{0\}) = \frac{\phi_\infty(1 - \nu\{0\})}{1 - c_N \phi_\infty} > \frac{1 + \kappa}{1 - c_N \phi_\infty}.$$

We will prove that q_N^+ is a bounded sequence. To this end, we will proceed by contradiction. Assume that there exists a sequence (n) such that $\lim_{n \rightarrow +\infty} q(n) = +\infty$. Since the sequence of functions ψ_N converge uniformly, one can extract a subsequence (p) from (n) such that: $c_{(p)} \rightarrow c^*$ and $\psi_{(p)}$ converge uniformly to ψ^* . Therefore, the sequence of functions $\left(f_p : q \mapsto \int_0^{+\infty} (1 - c_{(p)} \phi_\infty) \psi_{(p)}(qt) \nu(dt) \right)$ converge uniformly to $f^* : q \mapsto \int_0^{+\infty} (1 - c^* \phi_\infty) \psi^*(qt) \nu(dt)$. Hence,

$$\lim_{n \rightarrow +\infty} f_p(q_p) \rightarrow \lim_{x \rightarrow +\infty} f^*(x) = \psi_\infty(1 - c^* \phi_\infty)(1 - \nu\{0\}) = \phi_\infty(1 - \nu\{0\}) > 1 + \kappa,$$

which is in contradiction with the fact that:

$$\int_0^\infty (1 - c_N \phi_\infty) \psi_N(q_N^+ t) \nu(dt) = 1 + \kappa.$$

Now, consider η_N the unique solution of:

$$1 = \frac{\int_0^{+\infty} \psi_N(q_N^+ x) \nu(dx) F^{B_N}(dy)}{\int_0^{+\infty} \psi_N(q_N t) \frac{y+t}{t+\eta_N} \nu(dt)}.$$

Such η_N exists and is unique by Proposition 4.4. Set $q_i = q_N^+(\tau_i + \eta_N)$. We will prove that this choice of $q_j, j = 1, \dots, n$ guarantees:

$$\frac{h_j(q_1, \dots, q_n)}{q_j} \leq 1,$$

a.s. for n large enough. We have:

$$\begin{aligned} \frac{h_j(q_1, \dots, q_n)}{q_N^+(\tau_j + \eta_N)} &= \frac{1}{N(\tau_j + \eta_N)} y_j^* \left(\frac{1}{n} \sum_{i \neq j} \frac{\psi_N(q_N^+(\tau_i + \eta_N))}{\tau_i + \eta_N} y_i y_i^* \right)^{-1} y_j \\ &\leq \frac{1}{N(\tau_j + \eta_N) \int_0^{+\infty} \psi_N(q_N^+ x) \nu(dx)} y_j^* \left(\frac{1}{n} \sum_{i \neq j} \frac{\bar{\psi}_N(q_N^+ \tau_i)}{\tau_i + \eta_N} y_i y_i^* \right)^{-1} y_j, \end{aligned}$$

where $\bar{\psi}_{q_N^+}(x) = \frac{\psi_N(q_N^+ x)}{\int_0^{+\infty} \psi_N(q_N^+ t) \nu(dt)}$. From item *ii*) of Lemma 4.5, we have:

$$\max_{1 \leq j \leq n} \left| \frac{1}{N(\tau_j + \eta_N)} y_j^* \left(\frac{1}{n} \sum_{i \neq j} \frac{\bar{\psi}_{q_N^+}(q_N^+ \tau_i)}{\tau_i + \eta_N} y_i y_i^* \right)^{-1} y_j - \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\tau_j + \eta_N} T_N \right| \xrightarrow{\text{a.s.}} 0.$$

where $T_N = \left(\frac{1}{n} \sum_{i=1}^n \frac{\bar{\psi}_{q_N^+}(\tau_i)(B_N + \tau_i I_N)}{(\tau_i + \eta_N)(1 + e_i)} \right)^{-1}$ with e_1, \dots, e_n are the unique solutions to the following system of equations:

$$e_k = \frac{\bar{\psi}_{q_N^+}(\tau_k)}{n} \text{Tr} \frac{(B_N + \tau_k I_N)}{\tau_k + \eta_N} \left(\frac{1}{n} \sum_{i=1}^n \frac{\bar{\psi}_{q_N^+}(\tau_i)(B_N + \tau_i I_N)}{(\tau_i + \eta_N)(1 + e_i)} \right)^{-1}.$$

Let j_0 be the index of the maximum element in $\{e_1, \dots, e_n\}$. Then, there exists $\epsilon_n \downarrow 0$ such that for all $j = 1, \dots, n$

$$\frac{h_j(q_1, \dots, q_n)}{q_j} \leq \frac{1 + e_{j_0}}{\int_0^{+\infty} \psi_N(q_N^+ x) \nu(dx)} \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\tau_j + \eta_N} \left(\frac{1}{n} \sum_{i=1}^n \frac{\bar{\psi}_{q_N^+}(\tau_i)(B_N + \tau_i I_N)}{\tau_i + \eta_N} \right)^{-1} + \epsilon_n. \quad (4.13)$$

As $\left\| \bar{\psi}_{q_N^+} \right\|_{\infty} = \frac{\psi_{\infty}}{\int_0^{+\infty} \psi_N(q_N^+ t) \nu(dt)} = \frac{\phi_{\infty}}{1 + \kappa} \leq \frac{1}{c_N(1 + \kappa)}$, we obtain from item *iii*) of Lemma 4.5,

$$\max_{1 \leq k \leq n} e_k \leq \frac{c_N \left\| \bar{\psi}_{q_N^+} \right\|_{\infty}}{1 - c_N \left\| \bar{\psi}_{q_N^+} \right\|_{\infty}} + \epsilon'_n = \frac{c_N \phi_{\infty}}{1 + \kappa - c_N \phi_{\infty}} + |o(1)|. \quad (4.14)$$

where $o(1)$ refers to some sequences converging almost surely to zero as n grow to infinity. Plugging (4.14) into (4.13), and using the fact that:

$$\frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\tau_j + \eta_N} \left(\frac{1}{n} \sum_{i=1}^n \frac{\bar{\psi}_{q_N^+}(\tau_i)(B_N + \tau_i I_N)}{\tau_i + \eta_N} \right)^{-1} \leq 1 + |o(1)|,$$

we finally get:

$$\frac{h_j(q_1, \dots, q_n)}{q_j} \leq \frac{1 - c_N \phi_{\infty}}{1 + \kappa - c_N \phi_{\infty}} + |o(1)|$$

thereby establishing that:

$$h_j(q_1, \dots, q_n) < q_j$$

a.s. for n large enough.

Proof of Theorem 2.2: Asymptotic Convergence of the Robust-Scatter Estimator. The proof of Theorem 2.2 heavily relies on the new rewriting of the robust-scatter estimate as:

$$\hat{C}_N = \frac{1}{n} \sum_{i=1}^n v(q_i) y_i y_i^*, \quad (4.15)$$

where q_1, \dots, q_n are the unique solutions of the following system of equations:

$$q_j = y_j^* \left(\frac{1}{n} \sum_{i=1, i \neq j}^n v(q_i) y_i y_i^* \right)^{-1} y_j,$$

their existence and uniqueness in the asymptotic regime being established in the proof of Theorem 2.1. From the rewriting of \hat{C}_N in (4.15), it appears that an in-depth study of the asymptotic behaviour of q_1, \dots, q_n can be a good starting point. As mentioned in our heuristic analysis, one intuitively expects the q_1, \dots, q_n to approach in the asymptotic regime $\delta_1, \dots, \delta_n$, the solutions of the following system of equations:

$$\delta_i = \frac{1}{N} \text{Tr}(B_N + \tau_i I_N) \left(\frac{1}{n} \sum_{j=1}^n \frac{v(\delta_j)(B_N + \tau_j I_N)}{1 + c_N \psi_N(\delta_j)} \right)^{-1}.$$

This intuition underlies the proof of Theorem 2.2. In particular, we will prove that:

$$f_i = \frac{v(q_i)}{v(\delta_i)}, \quad i = 1, \dots, n$$

satisfy:

$$\max_{1 \leq i \leq n} |f_i - 1| \xrightarrow{\text{a.s.}} 0. \quad (4.16)$$

This in particular will allow us to state that \hat{C}_N can be approximated by $\hat{S}_N = \frac{1}{n} \sum_{i=1}^n v(\delta_i) y_i y_i^*$. The importance of this finding lies in the fact that unlike \hat{C}_N , \hat{S}_N follows a classical random matrix model, thereby opening up possibilities of exploiting an important load of available results. Prior to proceeding into the proof of the convergence stated in (4.16), we first need to introduce the following key lemmas that allow to identify the intervals within which lie almost surely quantities q_1, \dots, q_n and $\delta_1, \dots, \delta_n$. We start by handling terms $\delta_1, \dots, \delta_n$. We have in particular the following Lemma:

Lemma 4.6. *Let:*

$$h_j : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \\ (x_1, \dots, x_n) \mapsto \begin{cases} \frac{1}{N} \text{Tr}(B_N + \tau_j I) \left(\frac{1}{n} \sum_{i=1}^n \frac{\psi(x_i)(B_N + \tau_i I)}{x_i(1 + c_N \psi(x_i))} \right)^{-1} & \text{if } \exists x_i \neq 0 \\ \frac{1}{N} \text{Tr}(B_N + \tau_j I) \left(\frac{1}{n} \sum_{i=1}^n v(0)(B_N + \tau_i I) \right)^{-1} & \text{otherwise.} \end{cases}$$

Then, for all large n , there exists a unique vector $(\delta_1, \dots, \delta_n) \in \mathbb{R}_+^n$ such that:

$$h_j(\delta_1, \dots, \delta_n) = \delta_j, \quad \forall 1 \leq j \leq n. \quad (4.17)$$

Besides, vector $(\delta_1, \dots, \delta_n)$ is given by:

$$(\delta_1, \dots, \delta_n) = \lim_{t \rightarrow +\infty} (\delta_1^t, \dots, \delta_n^t),$$

with $(\delta_1^0, \dots, \delta_n^0)$ chosen arbitrarily in \mathbb{R}_+^n and:

$$\delta_j^{t+1} = h_j(\delta_1^t, \dots, \delta_n^t).$$

Moreover, there exists δ_N^+ and δ_N^- with $\limsup \delta_N^+ < \infty$ and $\liminf \delta_N^- > 0$ and η_N^+, η_N^- such that, almost surely for n large enough, we have:

$$\delta_N^-(\tau_j + \eta_N^-) \leq \delta_j \leq \delta_N^+(\tau_j + \eta_N^+), \quad 1 \leq j \leq n.$$

Moreover, η_N^+ and η_N^- satisfy:

$$\eta_N^+ = O\left(\frac{1}{N} \text{Tr } B_N\right) \text{ and } \eta_N^- = O\left(\frac{1}{N} \text{Tr } B_N\right).$$

Proof. Similar to the proof of Theorem 2.1, we can show along the same lines that $h = (h_1, \dots, h_n)$ is a standard interference function. It remains to prove the existence of $(\delta_1, \dots, \delta_n)$ such that $h_j(\delta_1, \dots, \delta_n) < \delta_j$. To this end, take $\delta_j = \xi(\tau_j + \eta_N)$, where η_N is the unique solution to the following equation:

$$\int \frac{F^{B_N}(dy)}{\int \frac{y+t}{t+x} \nu(dt)} = 1.$$

Such η_N exists according to proposition 4.4. Then:

$$\lim_{\xi \rightarrow +\infty} \frac{h_j(\delta_1, \dots, \delta_n)}{\delta_j} = \frac{1 + c_N \psi_\infty}{\psi_\infty} \frac{1}{N} \text{Tr} \frac{(B + \tau_j I)}{\tau_j + \eta_N} \left(\frac{1}{n} \sum_{i=1}^n \frac{B + \tau_i I}{\tau_i + \eta_N} \right)^{-1}. \quad (4.18)$$

Again, the limit in the above equation (4.18) can be controlled using proposition 4.4, thereby yielding:

$$\lim_{\xi \rightarrow +\infty} \frac{h_j(\delta_1, \dots, \delta_n)}{\delta_j} \leq \frac{1 + c_N \psi_\infty}{\psi_\infty} + \epsilon_n = \frac{1}{\phi_\infty} + \epsilon_n$$

where $\epsilon_n \downarrow 0$ almost surely. As $\phi_\infty > 1$, one can conclude that there exists $\delta_1, \dots, \delta_n$, such that for enough large N ,

$$h_j(\delta_1, \dots, \delta_n) < \delta_j.$$

We are now in position to prove the uniform boundedness of δ_i . For that, consider $\theta > 0$ such that $\theta < \frac{\phi_\infty}{2(1-c+\phi_\infty)}$. Let M_θ be chosen such that $\nu(M_\theta, +\infty) < \theta$ and M_θ is greater than the limit support of $\|B_N\|$. Set δ_N^- and δ_N^+ so that the following conditions are fulfilled:

$$\int_0^{+\infty} \frac{\psi_N(\delta_N^+ t)}{1 + c_N \psi(\delta_N^+ t)} \nu(dt) > 1, \quad (4.19)$$

$$\int_0^{M_\theta} \psi_N(\delta_N^-(t + M_\theta)) \nu(dt) < \frac{1}{2}. \quad (4.20)$$

Such choices are possible since

- $\lim_{\delta^+ \rightarrow +\infty} \int_0^{+\infty} \frac{\psi_N(\delta^+ t)}{1 + c_N \psi(\delta^+ t)} \nu(dt) = \frac{(1 - \nu\{0\})\psi_\infty}{1 + c_N \psi_\infty} = (1 - \nu\{0\})\phi_\infty > 1,$
- $\lim_{\delta^- \rightarrow 0^+} \int_0^{M_\theta} \psi(\delta^-(t + M_\theta)) \nu(dt) = 0,$

Moreover, we can check that one can choose δ_N^- and δ_N^+ such that $\liminf \delta_N^- > 0$ and $\limsup \delta_N^+ < +\infty$. As a matter of fact, building on the same reasoning used to show that $\limsup \alpha_N < +\infty$ in the proof of Theorem 2.1, we take δ_N^+ and δ_N^- the positive reals that verify:

$$\int_0^{+\infty} \frac{\psi(\delta_N^+ t)}{1 + c_N \psi(\delta_N^+ t)} \nu(dt) = 1 + \kappa$$

$$\int_0^{M_\theta} \psi(\delta_N^-(t + M_\theta)) \nu(dt) = \kappa,$$

where $0 \leq \kappa < \frac{1}{2}$ satisfies $1 + \kappa < (1 - \nu\{0\})\phi_\infty$. Assume that $\liminf \delta_N^- = 0$. There exists a sequence (n) such that $\lim_{n \rightarrow +\infty} \delta_N^- = 0$. Since the sequence of functions ψ_N converge uniformly, one can extract a subsequence (p) from (n) such that $c(p) \rightarrow c^*$ and ψ_p converge uniformly to ψ^* . Therefore, the sequence of functions $f_p : \alpha \mapsto \int_0^{M_\theta} \psi(\alpha(t + M_\theta)) \nu(dt)$ converge uniformly to $f^* : \alpha \mapsto \int_0^{M_\theta} \psi(\alpha(t + M_\theta)) \nu(dt)$. Hence:

$$\lim_{n \rightarrow +\infty} f_p(\delta_p^-) \rightarrow \lim_{x \rightarrow 0} f^*(x) = 0.$$

which is in contradiction with the fact that $f_p(\delta_p^-) = \kappa$. The same method can be used to prove that $\limsup \delta_N^+ < \infty$. Consider now the function $f^+ : t \mapsto \frac{\psi(\delta_N^+ t)}{(1 + c_N \psi(\delta_N^+ t))}$ in the domain $t \in [0, \infty)$. Define η_N^+ the unique solution to the following equation:

$$1 = \int \frac{F^{B_N}(dy) \int_0^{+\infty} f^+(x) \nu(dx)}{\int_0^{+\infty} \frac{y+t}{t+\eta_N^+} f^+(t) \nu(dt)}.$$

Similarly, define on \mathbb{R}^+ the function $f^- : t \mapsto \psi_\infty 1_{\{t \geq M_\theta\}} + \psi(2\delta^-(t + M_\theta)) 1_{\{t < M_\theta\}}$. Let η_N^- be the unique solution to the following equation:

$$1 = \int \frac{F^{B_N}(dy) \int_0^{+\infty} f^-(x) \nu(dx)}{\int_0^{+\infty} \frac{y+t}{t+\eta_N^-} f^-(t) \nu(dt)}. \quad (4.21)$$

Note that from proposition 4.4, η_N^+ and η_N^- are well-defined and satisfy:

$$\eta_N^+ = \mathcal{O}\left(\frac{1}{N} \text{Tr } B_N\right), \eta_N^- = \mathcal{O}\left(\frac{1}{N} \text{Tr } B_N\right).$$

Set for all i , $\delta_i^0 = \delta_N^+(\tau_i + \eta_N^+)$. Define recursively the sequences:

$$\delta_j^{t+1} = \frac{1}{N} \text{Tr} (B_N + \tau_j I_N) \left(\frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_i^t)(B_N + \tau_i I_N)}{\delta_i^t(1 + c_N \psi(\delta_i^t))} \right)^{-1}.$$

From the previous analysis, $\delta_i = \lim_{t \rightarrow +\infty} \delta_i^t$. To prove the uniform boundedness of δ_i , one can proceed by induction on t . For $t = 0$, the result is true. Let $t \in \mathbb{N}^*$ and assume that $\delta_j^k \leq \delta_N^+(\tau_j + \eta_N^+)$ holds true for any $k \leq t$ and $j = 1, \dots, n$. We propose to prove it for $k = t + 1$. We have

$$\frac{\delta_j^{t+1}}{\delta_N^+(\tau_j + \eta^+)} = \frac{1}{N} \text{Tr} \frac{B + \tau_j I}{\delta_N^+(\tau_j + \eta^+)} \left(\frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_i^t)(B + \tau_i I)}{\delta_i^t(1 + c_N \psi(\delta_i^t))} \right)^{-1}.$$

From the induction assumption along with the fact that $x \mapsto \frac{\psi(x)}{x(1+c_N\psi(x))}$ is non-increasing, we obtain:

$$\frac{\psi(\delta_i^t)}{\delta_i^t (1 + c_N \psi(\delta_i^t))} \geq \frac{\psi(\delta_N^+(\tau_i + \eta_N^+))}{\delta_N^+(\tau_i + \eta_N^+) (1 + c_N \psi(\delta_N^+(\tau_i + \eta_N^+)))}.$$

Hence,

$$\begin{aligned} \frac{\delta_j^{t+1}}{\delta_N^+(\tau_j + \eta_N^+)} &\leq \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\tau_j + \eta_N^+} \left(\frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_N^+ \tau_i)}{(1 + c_N \psi(\delta_N^+ \tau_i))} \frac{B_N + \tau_i I_N}{\tau_i + \eta_N^+} \right)^{-1} \\ &= \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\tau_j + \eta_N^+} \left(\frac{1}{n} \sum_{i=1}^n f^+(\tau_i) \frac{B_N + \tau_i I_N}{\tau_i + \eta_N^+} \right)^{-1}. \end{aligned}$$

From Remark 4.1 along with Lemma 4.3, function $t \mapsto \frac{f^+(t)}{\int f^+(x) \nu(dx)}$ satisfies the assumptions of proposition 4.4. We have therefore,

$$\frac{\int_0^{+\infty} f^+(x) \nu(dx)}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\tau_j + \eta_N^+} \left(\frac{1}{n} \sum_{i=1}^n f^+(\tau_i) \frac{B_N + \tau_i I_N}{\tau_i + \eta_N^+} \right)^{-1} \leq 1 + \epsilon_{n,j}, \quad \forall 1 \leq j \leq n.$$

where $\max_j |\epsilon_{n,j}|$ converges to zero almost surely. Equation (4.19) guarantees that $\int_0^{+\infty} f^+(x) \nu(dx) > 1$, thereby showing, that almost surely for n large enough:

$$\frac{\delta_j^{t+1}}{\delta_N^+(\tau_j + \eta_N^+)} \leq 1.$$

We will now prove the lower-bound inequality. Similarly, consider for all i , $\delta_i^0 = 2\delta_N^-(\tau_i + \eta_N^-)$. The sequence:

$$\delta_j^{t+1} = \frac{1}{N} \text{Tr}(B + \tau_j I) \left(\frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_i^t)(B + \tau_i I)}{\delta_i^t (1 + c_N \psi(\delta_i^t))} \right)^{-1}$$

converges to δ_i^* as $t \rightarrow +\infty$. In the same way as for the upper-bound inequality, we will show the result by induction on t . For $t = 0$, the result is true. Let $t \in \mathbb{N}^+$ and assume that $\delta_j^k \geq \delta_N^-(\tau_j + \eta_N^-)$ holds true for any $k \leq t$ and $j = 1, \dots, n$. We propose to prove the result

for $k = t + 1$. Similar to above, using the fact that $x \mapsto \frac{\psi(x)}{x}$ is non-increasing, we have:

$$\begin{aligned}
\frac{\delta_j^{t+1}}{\delta_N^-(\tau_j + \eta_N^-)} &= \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\delta_N^-(\tau_j + \eta_N^-)} \left(\frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_i^t)(B_N + \tau_i I_N)}{\delta_i^t(1 + c_N \psi(\delta_i^t))} \right)^{-1} \\
&\geq \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{(\tau_j + \eta_N^-)} \left(\sum_{i=1}^n \frac{\psi(\delta_N^-(\tau_i + \eta_N^-))(B_N + \tau_i I_N)}{(\tau_i + \eta_N^-)} \right)^{-1} \\
&= \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{(\tau_j + \eta_N^-)} \left(\sum_{\tau_i \geq M_\theta} \frac{\psi(\delta_N^-(\tau_i + \eta_N^-))(B_N + \tau_i I_N)}{(\tau_i + \eta_N^-)} + \sum_{\tau_i \leq M_\theta} \frac{\psi(\delta_N^-(\tau_i + \eta_N^-))(B_N + \tau_i I_N)}{(\tau_i + \eta_N^-)} \right)^{-1} \\
&\stackrel{(a)}{\geq} \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{(\tau_j + \eta_N^-)} \left(\sum_{\tau_i \geq M_\theta} \frac{\psi_\infty(B_N + \tau_i I_N)}{(\tau_i + \eta_N^-)} + \sum_{\tau_i \leq M_\theta} \frac{\psi(\delta_N^-(M_\theta + \tau_i))(B_N + \tau_i I_N)}{(\tau_i + \eta_N^-)} \right)^{-1} \\
&= \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{(\tau_j + \eta_N^-)} \left(\sum_{i=1}^n f^-(\tau_i) \frac{B_N + \tau_i I_N}{\tau_i + \eta_N^-} \right),
\end{aligned}$$

where (a) follows from the fact that $\eta_N^- \leq \|B_N\| \leq M_\theta$. Again, from remark 4.1 and Lemma 4.3, function $t \mapsto \frac{f^-(t)}{\int f^-(x)\nu(dx)}$ satisfies the assumptions of proposition 4.4. We have therefore,

$$\frac{\int_0^{+\infty} f^-(x)\nu(dx)}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\tau_j + \eta_N^-} \left(\frac{1}{n} \sum_{i=1}^n f^-(\tau_i) \frac{B_N + \tau_i I_N}{\tau_i + \eta_N^-} \right)^{-1} \geq 1 - |\epsilon_{n,j}|,$$

where $\max_j |\epsilon_{n,j}|$ converges to zero almost surely. On the other hand,

$$\int_0^{+\infty} f^-(x)\nu(dx) \leq \theta \frac{\phi_\infty}{1 - c_+ \phi_\infty} + \int_0^{M_\theta} \psi(\delta_N^-(t + M_\theta)) < 1,$$

and hence, almost surely, for enough large n ,

$$\frac{\delta_j^{t+1}}{\delta^-(\tau_j + \eta_N^-)} \geq \frac{1}{N} \text{Tr} \frac{B + \tau_j I}{(\tau_j + \eta_N^-)} \left(\sum_{i=1}^n f^-(\tau_i) \frac{B_N + \tau_i I_N}{\tau_i + \eta_N^-} \right) > 1.$$

□

The following refinement of Lemma 4.6 will be required in the proof of the asymptotic convergence of the robust-scatter estimator.

Lemma 4.7. *Let (κ, M_κ) be couples indexed by κ with $0 < \kappa < 1$ and $M_\kappa > 0$ such that $\nu(M_\kappa, +\infty) < \kappa$. Then, for sufficiently small κ the following system of equations:*

$$\delta_j = \frac{1}{N} \text{Tr} (B_N + \tau_j I_N) \left(\frac{1}{n} \sum_{\substack{i=1 \\ \tau_i \leq M_\kappa}}^n \frac{\psi(\delta_i)(B_N + \tau_i I_N)}{\delta_i(1 + c_N \psi(\delta_i))} \right)^{-1}, \quad \forall 1 \leq j \leq n \quad (4.22)$$

has a unique vector solution $(\delta_1^\kappa, \dots, \delta_n^\kappa)$ for all large n a.s, and there exists $\delta_N^{-, \kappa_0}, \delta_N^{+, \kappa_0}$ with $\limsup \delta_N^{+, \kappa_0} < \infty$ and $\liminf \delta_N^{-, \kappa_0} > 0$ and $\eta_N^{-, \kappa_0}, \eta_N^{+, \kappa_0}$ such that for all $\kappa < \kappa_0$ small:

$$\delta_N^{-, \kappa}(\tau_i + \eta_N^{-, \kappa}) \leq \delta_i^\kappa \leq \delta_N^{+, \kappa}(\tau_i + \eta_N^{+, \kappa}), \quad i = 1, \dots, n$$

for all large n a.s. Moreover, η_N^{+, κ_0} and η_N^{-, κ_0} satisfies:

$$\eta_N^{+, \kappa_0} = O\left(\frac{1}{N} \text{Tr } B_N\right) \quad \text{and} \quad \eta_N^{-, \kappa_0} = O\left(\frac{1}{N} \text{Tr } B_N\right)$$

Proof. The same proof as that of Lemma 4.7 holds by taking κ_0 smaller than θ and choosing δ_N^+ so that it satisfies:

$$\int_0^{M_{\kappa_0}} \frac{\psi(\delta_N^+ t)}{1 + c_N \psi(\delta_N^+ t)} \nu(dt) > 1$$

while δ_N^- is set in the same way as before. \square

We will now provide similar results for the random quantities q_1, \dots, q_n . In particular, we have the following Lemma:

Lemma 4.8. *Let $q_i \triangleq y_i^* \hat{C}_{(i)}^{-1} y_i, i = 1, \dots, n$. There exists $q_N^+, q_N^-, \alpha_N^+, \alpha_N^- > 0$ with $\limsup_N q_N^+ < +\infty$ and $\liminf_N q_N^- > 0$ such that, for all large n a.s.,*

$$q_N^-(\tau_i + \alpha_N^-) \leq q_i \leq q_N^+(\tau_i + \alpha_N^+), \quad i = 1, \dots, n. \quad (4.23)$$

Proof. The proof is based on the same tools as those used to show Lemma 4.6. The single difference is that the random quantities q_i involve quadratic forms which will be treated by resorting to Lemma 4.5. First recall that q_1, \dots, q_n are given by:

$$(q_1, \dots, q_n) = \lim_{t \rightarrow +\infty} (q_1^t, \dots, q_n^t)$$

with (q_1^0, \dots, q_n^0) chosen arbitrarily in \mathbb{R}_+^n and:

$$q_j^{t+1} = \frac{1}{N} y_j^* \left(\frac{1}{n} \sum_{i=1, i \neq j}^n \frac{\psi_N(q_i^t)}{q_i^t} y_i y_i^* \right)^{-1} y_j.$$

Similar to the proof of Theorem 2.1, consider (q_N^+) so that $\int_0^{+\infty} \psi_N(q_N^+ t) \nu(dt) > \frac{1}{1 - c_N \phi_\infty}$ and $\limsup q_N^+ < +\infty$. Let α_N^+ be the unique solution of:

$$1 = \int \frac{\int_0^{+\infty} \psi(q_N^+ x) \nu(dx)}{\int_0^{+\infty} \psi(q_N^+ t) \frac{y+t}{t+\alpha_N^+} \nu(dt)} F^{B_N}(dy).$$

Set $q_i^+ = \frac{q_N^+}{2}(\tau_i + \alpha_N^+)$. We will prove by induction on t that $q_i^t \leq q_N^+(\tau_i + \alpha_N^+)$. For $t = 0$, the result holds true. Assume now that for all $k \leq t$:

$$q_i^k \leq q_N^+(\tau_i + \alpha_N^+),$$

and let us show that $q_i^{t+1} \leq q_N^+(\tau_i + \alpha_N)$. We have:

$$\begin{aligned} \frac{q_j^{t+1}}{q_N^+(\tau_j + \alpha_N)} &= \frac{1}{Nq_N^+(\tau_j + \alpha_N)} y_j^* \left(\frac{1}{n} \sum_{i \neq j} \frac{\psi(q_i^t)}{q_i^t} y_i y_i^* \right)^{-1} y_j \\ &\leq \frac{1}{N(\tau_j + \alpha_N)} y_j^* \left(\frac{1}{n} \sum_{i \neq j} \frac{\psi(q_N^+ \tau_i)}{\tau_i + \alpha_N^+} y_i y_i^* \right)^{-1} y_j. \end{aligned}$$

Let $\bar{\psi}_{q_N^+}(x) = \frac{\psi(q_N^+ x)}{\int_0^{+\infty} \psi(q_N^+ t) \nu(dt)}$. From item *ii*) in Lemma 4.5, we have:

$$\max_{1 \leq j \leq n} \left| \frac{1}{N(\tau_j + \alpha_N) \int_0^{+\infty} \psi(q_N^+ x) \nu(dx)} y_j^* \left(\frac{1}{n} \sum_{i \neq j} \frac{\bar{\psi}_{q_N^+}(\tau_i)}{\tau_i + \alpha_N^+} y_i y_i^* \right)^{-1} y_j - \frac{1}{N \int_0^{+\infty} \psi(q_N^+ x) \nu(dx)} \text{Tr} \frac{B_N + \tau_j I_N}{\tau_j + \alpha_N^+} T_N \right| \xrightarrow{\text{a.s.}} 0, \quad (4.24)$$

where $T_N = \left(\frac{1}{n} \sum_{i=1}^n \frac{\bar{\psi}_{q_N^+}(\tau_i)(B_N + \tau_i I_N)}{(\tau_i + \alpha_N^+)(1 + e_i)} \right)^{-1}$ with e_1, \dots, e_n the unique solutions to the following system of equations:

$$e_k = \frac{\bar{\psi}_{q_N^+}(\tau_k)}{n} \text{Tr} \frac{B_N + \tau_k I_N}{\tau_k + \alpha_N^+} \left(\frac{1}{n} \sum_{i=1}^n \frac{\bar{\psi}_{q_N^+}(\tau_i)(B_N + \tau_i I_N)}{(\tau_i + \alpha_N^+)(1 + e_i)} \right)^{-1}, \quad k = 1, \dots, n.$$

The limit of the convergence in (4.24) can be bounded as:

$$\begin{aligned} \frac{1}{N \int_0^{+\infty} \psi(q_N^+ x) \nu(dx)} \text{Tr} \frac{B_N + \tau_j I_N}{\tau_j + \alpha_N^+} T_N &\leq (1 + \max_{1 \leq k \leq n} e_k) \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\tau_j + \alpha_N^+} \left(\frac{1}{n} \sum_{i=1}^n \frac{\bar{\psi}_{q_N^+}(\tau_i)(B_N + \tau_i I_N)}{\tau_i + \alpha_N^+} \right)^{-1} \\ &\stackrel{(a)}{\leq} \frac{1}{\int_0^{+\infty} \psi(q_N^+ x) \nu(dx)} (1 + \max_{1 \leq k \leq n} e_k) + \epsilon_{n,j} \end{aligned}$$

where $\max_{1 \leq j \leq n} |\epsilon_{n,j}|$ converges to zero almost surely (inequality (a) being a by-product of (4.3) in proposition 4.4). Finally, from item *iii*) of Lemma 4.5, we get:

$$\frac{1}{\int_0^{+\infty} \psi(q_N^+ x) \nu(dx)} \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\tau_j + \alpha_N^+} T_N \leq \frac{1}{\int_0^{+\infty} \psi(q_N^+ x) \nu(dx) - c_N \psi_\infty} + \epsilon'_{n,j}$$

with $\epsilon'_{n,j}$ converging to zero almost surely. Since q_N^+ satisfies:

$$\int_0^{+\infty} \psi(q_N^+ x) \nu(dx) > \frac{1}{1 - c_N \psi_\infty} = 1 + c_N \psi_\infty,$$

we obtain that:

$$\frac{q_j^{t+1}}{q_N^+(\tau_j + \alpha_N)} < 1$$

for n large enough a.s. In order to prove the lower bound in (4.23), the same reasoning as the one used in the previous lemma applies. In particular, it suffices to set $\theta > 0$ and M_θ such that $\theta < \frac{\phi_\infty}{2(1-c_+ \infty)}$, $\nu(M_\theta, +\infty) < \theta$ and $M_\theta \geq \|B_N\|$. Taking q_N^- such that:

$$\int_0^{M_\theta} \psi_N(q_N^-(t + M_\theta)) \nu(dt) < \frac{1}{2}$$

and setting α_N^- to the unique solution of the following equation:

$$1 = \int \frac{\int_0^{+\infty} f^-(x) \nu(dx)}{\int_0^{+\infty} \frac{y+t}{t+\alpha_N^-} f^-(t) \nu(dt)} F^{B_N}(dy)$$

with $f^- : t \mapsto \psi_\infty 1_{t \geq M_\theta} + \psi(2q_N^-(t + M_\theta)) 1_{t \leq M_\theta}$, we can establish by induction on t and using the same steps as in the control of the lower bound of δ_i that:

$$\frac{q_i}{q_N^-(\tau_i + \alpha_N^-)} > 1, \quad i = 1, \dots, n.$$

□

The determination of an interval in which lies all quantities $\delta_1, \dots, \delta_n$ is of utmost important in that it allows us to control the quadratic forms: $\frac{\psi(\delta_j)}{N\delta_j} y_j^* \left(\frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_i)}{\delta_i} y_i y_i^* \right)^{-1} y_j$ and $\frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* \left(\frac{1}{n} \sum_{i=1, \tau_i \leq M_\kappa} \frac{\psi(\delta_i^\kappa)}{\delta_i^\kappa} y_i y_i^* \right)^{-1} y_j$, where $\delta_1, \dots, \delta_n$ and $\delta_1^\kappa, \dots, \delta_n^\kappa$ are solutions of equations (4.22) and (4.17). In particular, we have the following two lemmas which easily follows from item-*i*) of Lemma 4.5.

Lemma 4.9. *Let Assumptions 1-4 hold true. Then,*

$$\max_{1 \leq j \leq n} \left| \frac{\psi(\delta_j)}{N\delta_j} y_j^* \left(\sum_{i=1, i \neq j}^n \frac{\psi(\delta_i)}{\delta_i} y_i y_i^* \right)^{-1} y_j - \psi(\delta_j) \right| \xrightarrow{\text{a.s.}} 0.$$

Lemma 4.10. *Let (κ, M_κ) be couples indexed by κ with $0 < \kappa < 1$, and $M_\kappa > 0$ such that $\nu(M_\kappa, \infty) < \kappa$. Then, for all $\kappa < \kappa_0$, we have:*

$$\max_{1 \leq j \leq n} \left| \frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* \left(\sum_{i=1, i \neq j}^n \frac{\psi(\delta_i^\kappa)}{\delta_i^\kappa} y_i y_i^* \right)^{-1} y_j - \psi(\delta_j^\kappa) \right| \xrightarrow{\text{a.s.}} 0,$$

where $\delta_i^\kappa, i = 1, \dots, n$ are defined the solutions of (4.22).

With these results at hand, we are now in position to prove $f_i = \frac{v(q_i)}{v(\delta_i)}$ satisfies:

$$\max_{1 \leq i \leq n} |f_i - 1| \xrightarrow{\text{a.s.}} 0.$$

As in [8], we will distinguish two cases: the case where all τ_i s are bounded and that of unbounded τ_i . The proof is merely based on the same techniques with only some modifications and will be detailed for sake of completeness.

All τ_i -s are Bounded. Assume that there exists a constant M such that $\tau_i \leq M$ for all $i = 1, \dots, n$. Define $f_i = \frac{v(q_i)}{v(\delta_i)} > 0$. Without loss of generality, we assume that $f_1 \leq \dots \leq f_n$.

We have:

$$f_j = \frac{v(\frac{1}{N}y_j^* \hat{C}_j^{-1} y_j)}{v(\delta_j)} \quad (4.25)$$

$$\begin{aligned} &= \frac{v\left(\frac{1}{N}y_j^* \left(\frac{1}{n} \sum_{i \neq j} v(q_i) y_i y_i^*\right)^{-1} y_j\right)}{v(\delta_j)} \\ &= \frac{v\left(\frac{1}{N}y_j^* \left(\frac{1}{n} \sum_{i \neq j} f_i v(\delta_i) y_i y_i^*\right)^{-1} y_j\right)}{v(\delta_j)} \\ &\leq \frac{v\left(\frac{1}{N}y_j^* \left(\frac{1}{n} \sum_{i \neq j} f_n v(\delta_i) y_i y_i^*\right)^{-1} y_j\right)}{v(\delta_j)} \\ &= \frac{v\left(\frac{\delta_j}{f_n \psi(\delta_j)} \frac{\psi(\delta_j)}{N \delta_j} \left(\frac{1}{n} \sum_{i \neq j} v(\delta_i) y_i y_i^*\right)^{-1} y_j\right)}{v(\delta_j)}. \end{aligned} \quad (4.26)$$

In a similar way, we also have:

$$f_1 \geq \frac{v\left(\frac{\delta_j}{f_1 \psi(\delta_j)} \frac{\psi(\delta_j)}{N \delta_j} \left(\frac{1}{n} \sum_{i \neq j} v(\delta_i) y_i y_i^*\right)^{-1} y_j\right)}{v(\delta_j)},$$

From Lemma 4.9, let $0 < \epsilon_n < 1$ with $\epsilon_n \downarrow 0$ such that for all large n , a.s. and for all $1 \leq j \leq n$:

$$\psi(\delta_j) - \epsilon_n < \frac{\psi(\delta_j)}{N \delta_j} y_j^* \left(\frac{1}{n} \sum_{i \neq j} v(\delta_i) y_i y_i^*\right)^{-1} y_j \leq \psi(\delta_j) + \epsilon_n. \quad (4.27)$$

In particular, since v is non-increasing, taking $j = n$ in (4.26) and applying the left-hand inequality in (4.27), we obtain:

$$f_n < \frac{v\left(\frac{\delta_n}{f_n \psi(\delta_n)} \max(\psi(\delta_n) - \epsilon_n, 0)\right)}{v(\delta_n)}. \quad (4.28)$$

Assume now that for some $\ell > 0$, $f_n > 1 + \ell$ infinitely often. Therefore, there exists a sequence (n) over which $f_{(n)} > 1 + \ell$ for n large enough. We distinguish two cases. First, assume that $\liminf \delta_{(n)} = 0$. There exists a sequence (p) obtained from a subsequence of (n) over which $\lim_{n \rightarrow +\infty} \delta_{(p)} = 0$.

From (4.28), we have:

$$\lim_{n \rightarrow +\infty} f_{(p)} \leq \lim_{n \rightarrow +\infty} \frac{v\left(\frac{1}{f_{(p)} v(\delta_{(p)})} \max(\psi(\delta_{(p)}) - \epsilon_{(p)}, 0)\right)}{v(\delta_{(p)})} = 1.$$

which is in contradiction with $f_{(p)} > 1 + \ell$. Therefore, for (4.28) to hold, we must have $\liminf \delta_n > \delta_{\min}$. Since all τ_i -s are bounded, $(\delta_n)_n$ is also a bounded sequence. One can thus extract a subsequence (q) extracted from (p) over which $\delta_{(q)} \rightarrow x > 0$ and $c_N \rightarrow c$. Let

$\psi_c(x) = \lim_{c_N \rightarrow c} \psi(x)$ and write (4.28) in the following equivalent form:

$$\left(1 - \frac{\epsilon(q)}{\psi(\delta(q))}\right) \frac{\psi(\delta(q))}{\psi\left(\frac{\delta(q)}{f(q)} \left(1 - \frac{\epsilon(q)}{\psi(\delta(q))}\right)\right)} < 1. \quad (4.29)$$

We therefore have:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left(1 - \frac{\epsilon(q)}{\psi(\delta(q))}\right) \frac{\psi(\delta(q))}{\psi\left(\frac{\delta(q)}{f(q)} \left(1 - \frac{\epsilon(q)}{\psi(\delta(q))}\right)\right)} &\geq \lim_{n \rightarrow +\infty} \left(1 - \frac{\epsilon(q)}{\psi(\delta(q))}\right) \frac{\psi(\delta(q))}{\psi\left(\frac{\psi(\delta(q))}{\ell+1} \left(1 - \frac{\epsilon(q)}{\psi(\delta(q))}\right)\right)} \\ &= \frac{\psi_c(x)}{\psi_c((1+\ell)^{-1}x)} > 1, \end{aligned}$$

which is in contradiction with (4.29). Symmetrically, we obtain that for $\epsilon_n \downarrow 0$ and for large n a.s.,

$$f_1 > \frac{v\left(\frac{\delta_1}{f_1 \psi(\delta_1)}\right)(\psi(\delta_1) + \epsilon_n)}{v(\delta_1)}$$

which is equivalent to:

$$\frac{f_1 v(\delta_1)}{v\left(\frac{\delta_1}{f_1 \psi(\delta_1)}(\psi(\delta_1) + \epsilon_n)\right)}.$$

We conclude using the same reasoning as above that for each $\ell > 0$ small $f_1 \geq 1 - \ell$ for all large n . a.s. so that finally, we have:

$$\max_{1 \leq i \leq n} |f_i - 1| \xrightarrow{\text{a.s.}} 0.$$

The uniform boundedness of τ_i implies that of q_i and δ_i , thereby ensuring that:

$$\max_{1 \leq i \leq n} |v(q_i) - v(\delta_i)| \xrightarrow{\text{a.s.}} 0.$$

Hence, for any $\ell > 0$, arbitrarily small, we have for all large n ,

$$(1 - \ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_i)}{\delta_i} y_i y_i^* \preceq \frac{1}{n} \sum_{i=1}^n v(q_i) y_i y_i^* \preceq (1 + \ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_i)}{\delta_i} y_i y_i^*,$$

Since the spectral norm of $\frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_i)}{\delta_i} y_i y_i^*$ is almost surely bounded and ℓ is arbitrary, we conclude that:

$$\left\| \hat{C}_N - \hat{S}_N \right\| \xrightarrow{\text{a.s.}} 0.$$

Unbounded τ_i . We now relax the boundedness assumption on the support of the distribution of τ_i . We will follow the same technique used in [8]. Similarly to [8], let (κ, M_κ) be couples indexed by κ such that for all large n , we have $\nu_n(M_\kappa, +\infty) < \kappa \leq \kappa_0$, for κ_0 small enough, and $M_\kappa \geq \limsup \|B_N\|$. Denote by $\mathcal{C}_\kappa = \{i, \tau_i \leq M_\kappa\}$ with cardinality $|\mathcal{C}_\kappa|$. Then,

$$\frac{|\mathcal{C}_\kappa|}{n} = 1 - \nu_n(M_\kappa, \infty) \geq 1 - \kappa.$$

In the sequel, we will differentiate the indexes in \mathcal{C}_κ from those in \mathcal{C}_κ^c . Define $f_1^\kappa, \dots, f_n^\kappa$ as:

$$f_i^\kappa = \frac{v(q_i)}{v(\delta_i^\kappa)},$$

where $\delta_1^\kappa, \dots, \delta_n^\kappa$ are the solutions of the system of equations (4.22) given in Lemma 4.7. Let $j \in \mathcal{C}_\kappa$ and denote by $f_1^\kappa = \min_{i \in \mathcal{C}_\kappa} f_i^\kappa$ and $f_n^\kappa = \max_{i \in \mathcal{C}_\kappa} f_i^\kappa$. We have:

$$\begin{aligned} f_j^\kappa &= \frac{v(q_j)}{v(\delta_j^\kappa)} \\ &= \frac{v\left(\frac{1}{N}y_j^* \left(\frac{1}{n} \sum_{i \neq j, i \in \mathcal{C}_\kappa} f_i^\kappa v(\delta_i^\kappa) y_i y_i^* + \frac{1}{n} \sum_{i \in \mathcal{C}_\kappa^c} v(q_i) y_i y_i^*\right)^{-1} y_j\right)}{v(\delta_j^\kappa)} \\ &\leq \frac{v\left(\frac{1}{N}y_j^* \left(\frac{1}{n} \sum_{i \neq j, i \in \mathcal{C}_\kappa} f_n^\kappa v(\delta_i^\kappa) y_i y_i^* + \frac{\psi_\infty}{nq_N} \sum_{i \in \mathcal{C}_\kappa^c} \frac{y_i y_i^*}{\tau_i + \alpha_N}\right)^{-1} y_j\right)}{v(\delta_j^\kappa)} \\ &= \frac{v\left(\frac{1}{Nf_n^\kappa} y_j^* \left(\frac{1}{n} \sum_{i \neq j, i \in \mathcal{C}_\kappa} v(\delta_i^\kappa) y_i y_i^* + \frac{\psi_\infty}{nq_N} \sum_{i \in \mathcal{C}_\kappa^c} \frac{y_i y_i^*}{\tau_i + \alpha_N}\right)^{-1} y_j\right)}{v(\delta_j^\kappa)}, \end{aligned}$$

where we used in the first inequality the fact that $q_i \geq q_N^-(\tau_i + \alpha_N^-)$. Since $f_n^\kappa = \frac{v(q_n)}{v(\delta_n^\kappa)} = \frac{\psi(q_n)}{\psi(\delta_n^\kappa)} \frac{\delta_n^\kappa}{q_n}$, we obtain:

$$\frac{v(q_N^+(\tau_n + \alpha_N^+))}{v(\delta_N^{+, \kappa_0}(\tau_n + \eta_N^{+, \kappa_0}))} \leq f_n^\kappa \leq \frac{\psi(q_N^+(\tau_n + \alpha_N^+)) \delta_N^{+, \kappa_0}(\tau_n + \eta_N^{+, \kappa_0})}{\psi(\delta_N^{+, \kappa_0}(\tau_n + \eta_N^{+, \kappa_0})) q_N^-(\tau_n + \alpha_N^-)}.$$

The above inequalities imply that $f_{\bar{n}}$ is almost surely bounded irrespective of κ small enough. To see that, note that if $\liminf \tau_{\bar{n}} = 0$, the left inequality ensures that $\liminf f_{\bar{n}}^\kappa > 0$ while if $\limsup_n \tau_{\bar{n}} = \infty$, the second inequality ensures that $\limsup f_{\bar{n}}^\kappa < \infty$. As a consequence, we can assume that $f_{\bar{n}}^\kappa > f_-$ for all large n and for all κ small enough. From this observation, for all large n , a.s. we have:

$$\begin{aligned} f_j^\kappa &\leq \frac{v\left(\frac{1}{Nf_{\bar{n}}^\kappa} y_j^* \left(\frac{1}{n} \sum_{i \neq j, i \in \mathcal{C}_\kappa} v(\delta_i^\kappa) y_i y_i^* + \frac{\psi_\infty}{nq_- f_-} \sum_{i \in \mathcal{C}_\kappa^c} \frac{1}{\tau_i + \alpha_N^-} y_i y_i^*\right)^{-1} y_j\right)}{v(\delta_j^\kappa)} \\ &= \frac{v\left(\frac{\delta_j^\kappa}{\psi(\delta_j^\kappa) f_{\bar{n}}^\kappa} \left[\frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* \left(\frac{1}{n} \sum_{i \neq j, i \in \mathcal{C}_\kappa} v(\delta_i^\kappa) y_i y_i^*\right)^{-1} y_j + w_{j,n}\right]\right)}{v(\delta_j^\kappa)}, \end{aligned} \quad (4.30)$$

where we defined similarly to [8] $w_{j,n}$ as:

$$w_{j,n} = \frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* (D_{\kappa,j} + C_\kappa)^{-1} y_j - \frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* D_{\kappa,j}^{-1} y_j$$

with

$$D_{\kappa,j} \triangleq \frac{1}{n} \sum_{i \in \mathcal{C}_\kappa, i \neq j} v(\delta_i^\kappa) y_i y_i^*, \quad C_\kappa = \frac{\psi_\infty}{nq_N^- f_-} \sum_{i \in \mathcal{C}_\kappa^c} \frac{1}{\tau_i + \alpha_N^-} y_i y_i^*$$

Using the resolvent identity $D^{-1} - F^{-1} = D^{-1}(F - D)F^{-1}$ (for any invertible matrices D and F) along with Cauchy-Schwartz inequality, we obtain:

$$|w_{n,j}| \leq \sqrt{\frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* (D_{\kappa,j} + C_\kappa)^{-1} C_\kappa (D_{\kappa,j} + C_\kappa)^{-1} y_j} \sqrt{\frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* D_{\kappa,j}^{-1} C_\kappa D_{\kappa,j}^{-1} y_j}.$$

Note that for κ small enough, matrix $D_{\kappa,j}$ is invertible. Besides, from assumption 3-ii), for κ small enough and for enough large n , $\nu_n([m, M_\kappa]) \geq c_+$. Using Lemma .14 in Appendix 4.2.2, we conclude that there exists C_1 such that $\min_j \lambda_1(D_{\kappa,j}) \geq C_1$. Since matrix C_κ has a bounded spectral norm, Theorem .15 in Appendix 4.2.2 along with the rank-1 perturbation Lemma [15, Lemma 2.6] yields:

$$\max_{j \in \mathcal{C}_\kappa} \left| \frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* D_{\kappa,j}^{-1} C_\kappa D_{\kappa,j}^{-1} y_j - \frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} \text{Tr}(B_N + \tau_j I_N) D_\kappa^{-1} C_\kappa D_\kappa^{-1} \right| \xrightarrow{\text{a.s.}} 0,$$

where $D_\kappa = D_{\kappa,j} + \frac{1}{n} v(\delta_n^\kappa) y_j y_j^*$. From $|\text{Tr} XY| \leq \|X\| \text{Tr} Y$ for positive definite Y , we have:

$$\frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} \text{Tr}(B_N + \tau_j I_N) D_\kappa^{-1} C_\kappa D_\kappa^{-1} \leq \frac{\|B_N + \tau_j I_N\|}{C_1^2 \delta_N^{+, \kappa_0} \psi_\infty(\tau_j + \eta_N^{-, \kappa_0})} \frac{1}{N} \sum_{i \in \mathcal{C}_\kappa} \frac{y_i^* y_i}{\tau_i + \alpha_N^-}$$

where

$$\frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} \frac{1}{N} \frac{y_i^* y_i}{\tau_i + \alpha_N^-} - \frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} \frac{1}{N} \text{Tr} \frac{B_N + \tau_i I_N}{\tau_i + \alpha_N^-} \xrightarrow{\text{a.s.}} 0.$$

Since $\frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} \frac{1}{N} \text{Tr} \frac{B_N + \tau_i I_N}{\tau_i + \alpha_N^-} \leq \frac{\|B_N\| + M_{\kappa_0}}{\alpha_N^- + M_{\kappa_0}} \frac{\nu_n(M_\kappa, \infty)}{n}$ for all $\kappa \leq \kappa_0$ and for all large n a.s., we have:

$$\max_{j \in \mathcal{C}_\kappa} \frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* D_{\kappa,j}^{-1} C_\kappa D_{\kappa,j}^{-1} y_j \leq K_1 \nu_n(M_\kappa, +\infty),$$

where K_1 is a constant that does not depend on $\kappa \leq \kappa_0$. In the same way, we can control the term $\frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* (D_{\kappa,j} + C_\kappa)^{-1} C_\kappa (D_{\kappa,j} + C_\kappa)^{-1} y_j$. Finally, we conclude that:

$$\max_{j \in \mathcal{C}_\kappa} |w_{j,n}| \leq K \nu_n(M_\kappa, +\infty) \quad (4.31)$$

for some constant K independent of $\kappa \leq \kappa_0$. Quantities $w_{j,n}$ being controlled for $j \in \mathcal{C}_\kappa$, we can now proceed in a similar way as in the case of the bounded τ_i case. Lemma ?? implies that for any fixed $\kappa > 0$ there exists a sequence $\epsilon_n^\kappa \downarrow 0$ such that a.s. for n large enough,

$$\max_{j \in \mathcal{C}_\kappa} \left| \frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* \left(\frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} \frac{\psi(\delta_i^\kappa)}{\delta_i^\kappa} y_i y_i^* \right)^{-1} y_j - \psi(\delta_j^\kappa) \right| \leq \epsilon_n^\kappa. \quad (4.32)$$

Combining (4.30), (4.31) and (4.32), we then have for all large n a.s. and for all $j \in \mathcal{C}_\kappa$,

$$f_j^\kappa \leq \frac{v \left[\frac{\delta_j^\kappa}{\psi(\delta_j^\kappa) f_n^\kappa} \max \left(\psi(\delta_j^\kappa) - \epsilon_n^\kappa - K \nu_n(M_\kappa, \infty), 0 \right) \right]}{v(\delta_j^\kappa)}$$

which for $j = \bar{n}$ becomes:

$$f_{\bar{n}}^\kappa \leq \frac{v \left[\frac{\delta_{\bar{n}}^\kappa}{\psi(\delta_{\bar{n}}^\kappa) f_{\bar{n}}^\kappa} \max \left(\psi(\delta_{\bar{n}}^\kappa) - \epsilon_n^\kappa - K \nu_n(M_\kappa, \infty), 0 \right) \right]}{v(\delta_{\bar{n}}^\kappa)}. \quad (4.33)$$

Assume that $\limsup f_{\bar{n}} > 1 + \ell$ for some $\ell > 0$. Let us restrict the sequence $f_{\bar{n}}$ to those indexes for which $f_{\bar{n}} > 1 + \ell$. Similar to the case of bounded τ_i , we can see that (4.33) implies that $\liminf \delta_{\bar{n}}^\kappa > \delta_{\min}$, a bound which can be chosen independent of $\kappa \leq \kappa_0$. In effect, from (4.33). we have:

$$f_{\bar{n}}^\kappa \leq \frac{v(0)}{v(\delta_{\bar{n}}^\kappa)}$$

which is equivalent to:

$$v(\delta_n^\kappa) \leq \frac{v(0)}{\ell + 1},$$

or also:

$$\delta_n^\kappa \geq v^{-1} \left(\frac{v(0)}{\ell + 1} \right)$$

Using the definition of ψ , (4.33) reads for κ sufficiently small:

$$\frac{\psi \left(\frac{\delta_n^\kappa}{f_n^\kappa} \left(1 - \frac{\epsilon_n^\kappa}{\psi(\delta_n^\kappa) - \frac{K\nu_n(M_\kappa, \infty)}{\psi(\delta_n^\kappa)}} \right) \right)}{\psi(\delta_n^\kappa) \left(1 - \frac{\epsilon_n^\kappa}{\psi(\delta_n^\kappa)} - \frac{K\nu_n(M_\kappa, \infty)}{\psi(\delta_n^\kappa)} \right)} \geq 1$$

or also for n large enough:

$$\frac{\psi(\delta_n^\kappa) \left(1 - \frac{\epsilon_n^\kappa}{\psi(\delta_n^\kappa)} - \frac{K\nu_n(M_\kappa, \infty)}{\psi(\delta_n^\kappa)} \right)}{\psi((f_n^\kappa)^{-1} \delta_n^\kappa)} \leq 1.$$

Hence,

$$\frac{\psi(\delta_n^\kappa) - \psi((f_n^\kappa)^{-1} \delta_n^\kappa)}{\psi((f_n^\kappa)^{-1} \delta_n^\kappa)} \left(1 - \frac{\epsilon_n^\kappa}{\psi(\delta_n^\kappa)} - \frac{K\nu_n(M_\kappa, \infty)}{\psi(\delta_n^\kappa)} \right) \leq \frac{\epsilon_n^\kappa}{\psi(\delta_n^\kappa)} + \frac{K\nu_n(M_\kappa, \infty)}{\psi(\delta_n^\kappa)}.$$

or equivalently:

$$\frac{\psi(\delta_n^{*,\kappa}) - \psi((f_n^{*,\kappa})^{-1} \delta_n^\kappa)}{\epsilon_n^\kappa + K\nu_n(M_\kappa, +\infty)} \leq \frac{\psi((f_n^\kappa)^{-1} \delta_n^{*,\kappa})}{\psi(\delta_n^{*,\kappa}) \left(1 - \frac{\epsilon_n^\kappa}{\psi(\delta_n^{*,\kappa})} - \frac{K\nu_n(M_\kappa, +\infty)}{\psi(\delta_n^{*,\kappa})} \right)}.$$

Since $f_n^\kappa > 1 + \ell$, $\psi((f_n^\kappa)^{-1} \delta_n^\kappa) < \psi(\delta_n^\kappa)$. Therefore, for κ chosen sufficiently small so that: $1 - \frac{\epsilon_n^\kappa}{\psi(\delta_n^\kappa)} - \frac{K\nu_n(M_\kappa, \infty)}{\psi(\delta_n^\kappa)} \leq \frac{1}{2}$, we have:

$$\frac{\psi(\delta_n^{*,\kappa}) - \psi((f_n^{*,\kappa})^{-1} \delta_n^\kappa)}{\epsilon_n^\kappa + K\nu_n(M_\kappa, +\infty)} \leq 2. \quad (4.34)$$

Since δ_n^κ belongs to the interval $[\delta_{\min}, \limsup \delta_N^+(M_\kappa) + \eta_N^+]$ for n large enough, taking the limit of (4.34) over some subsequences over which $\delta_n^\kappa \rightarrow x_\kappa \in [\delta_{\min}, \limsup \delta_N^+(M_\kappa) + \eta_N^+]$, $c_N \rightarrow c$ and $\nu_n(M_\kappa, \infty)$ converges, we obtain:

$$\frac{\psi_c(x_\kappa) - \psi_c(\frac{x_\kappa}{1+\ell})}{\lim_n \nu_n(M_\kappa, \infty)} \leq 2, \quad (4.35)$$

where $\psi_c = \lim_{c_N \rightarrow c} \psi_N$. We now operate on κ . If $\limsup_{\kappa \rightarrow 0} x_\kappa < \infty$, the left-hand side of (4.35) goes to $+\infty$ as $\kappa \rightarrow 0$ so that starting from κ sufficiently small and taking the limit over n on the considered subsequence raises a contradiction. If instead $\limsup_{\kappa \rightarrow 0} x_\kappa = +\infty$, then since $x_\kappa \leq 2 \limsup \delta_N^+ M_\kappa$, we have:

$$\frac{\psi_c(x_\kappa) - \psi_c(\frac{x_\kappa}{1+\ell})}{\lim_n \nu_n(M_\kappa, \infty)} \geq \frac{\psi_c(x_\kappa) - \psi_c(\frac{x_\kappa}{1+\ell})}{\lim_n \nu_n(\frac{x_\kappa}{\limsup \delta_N^+}, \infty)}.$$

Let $y_\kappa = g_c^{-1}(x_\kappa)$ with $g_c(x) = \frac{x}{1-c\phi(x)}$. Recall that $\psi_c = \frac{\phi \circ g_c^{-1}}{1-c\phi \circ g_c^{-1}}$. Then,

$$\begin{aligned} \frac{\psi_c(x_\kappa) - \psi_c(\frac{x_\kappa}{\ell+1})}{\lim_n \nu_n \left(\frac{x_\kappa}{\limsup \delta_N^+}, \infty \right)} &= \frac{\phi \circ g_c^{-1}(x_\kappa) - \phi \circ g_c^{-1}(\frac{x_\kappa}{\ell+1})}{(1-c\phi \circ g_c^{-1}(x_\kappa))(1-c\phi \circ g_c^{-1}(\frac{x_\kappa}{\ell+1})) \lim_n \nu_n \left(\frac{x_\kappa}{\limsup \delta_N^+}, \infty \right)} \\ &\geq \frac{\phi(y_\kappa) - \phi \circ g_c^{-1}(\frac{g_c(y_\kappa)}{\ell+1})}{\lim_n \nu_n \left(\frac{x_\kappa}{\limsup \delta_N^+}, \infty \right)} \\ &\geq \frac{\phi(y_\kappa) - \phi \circ g_c^{-1}(\frac{y_\kappa}{(\ell+1)(1-c+\phi_\infty)})}{\lim_n \nu_n \left(\frac{x_\kappa}{\limsup \delta_N^+}, \infty \right)}. \end{aligned}$$

Since $y_\kappa \rightarrow \infty$ as $x_\kappa \rightarrow \infty$, from Assumption 5, the right-hand side must go to ∞ as $x_\kappa \rightarrow \infty$. Therefore, taking κ sufficiently small and then consider the limsup over n on the subsequence under consideration raises a contradiction. Consequently, we must have $\limsup f_n^\kappa \leq 1 + \ell$ a.s. A similar reasoning allows to show that $\liminf f_1^\kappa \geq 1 - \ell$ a.s. for any given $\ell > 0$. We conclude thus:

$$\max_{j \in \mathcal{C}_\kappa} |f_j^\kappa - 1| \xrightarrow{\text{a.s.}} 0.$$

We will now deal with f_j^κ for $j \in \mathcal{C}_\kappa^c$. Recall that f_j is given by:

$$\begin{aligned} f_j &= \frac{v \left(\frac{1}{N} y_j^* \left(\frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} f_j^\kappa v(\delta_j^\kappa) y_i y_i^* + \frac{1}{n} \sum_{i \in \mathcal{C}^c, i \neq j} v(q_i) y_i y_i^* \right)^{-1} y_j \right)}{v(\delta_j^\kappa)} \\ &= \frac{v \left(\frac{\delta_j^\kappa}{\psi(\delta_j^\kappa)} \frac{\psi(\delta_j^\kappa)}{\delta_j^\kappa} \frac{1}{N} y_j^* \left(\frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} f_j^\kappa v(\delta_j^\kappa) y_i y_i^* + \frac{1}{n} \sum_{i \in \mathcal{C}^c, i \neq j} v(q_i) y_i y_i^* \right)^{-1} y_j \right)}{v(\delta_j^\kappa)} \\ &= \frac{v \left(\frac{\delta_j^\kappa}{\psi(\delta_j^\kappa)} \left[\frac{\psi(\delta_j^\kappa)}{\delta_j^\kappa} \frac{1}{N} y_j^* \left(\frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} f_i^\kappa v(\delta_i^\kappa) y_i y_i^* \right)^{-1} y_j + \tilde{w}_{j,n} \right] \right)}{v(\delta_j^\kappa)}, \end{aligned}$$

where

$$\tilde{w}_{j,n} = \frac{\psi(\delta_j^\kappa)}{N \delta_j^\kappa} y_j^* \left(\tilde{D}_\kappa + \tilde{C}_{\kappa,j} \right)^{-1} y_j - \frac{\psi(\delta_j^\kappa)}{N \delta_j^\kappa} y_j^* \tilde{D}_\kappa^{-1} y_j$$

with:

$$\begin{aligned} \tilde{D}_\kappa &= \frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} v(\delta_i^\kappa) y_i y_i^* \\ \tilde{C}_{\kappa,j} &= \frac{1}{n} \sum_{\substack{i \in \mathcal{C}_\kappa \\ i \neq j}} v(\delta_i^\kappa) y_i y_i^*. \end{aligned}$$

Using the same reasoning as with $w_{j,n}$ we can show that for κ sufficiently small and n large enough,

$$\max_{j \in \mathcal{C}_\kappa^c} |\tilde{w}_{j,n}| \leq K \nu_n(M_\kappa, \infty) \leq K \kappa$$

with K independent of $\kappa \leq \kappa_0$. On the other hand, since $\max_{i \in \mathcal{C}_\kappa^c} |f_i^\kappa - 1| \xrightarrow{\text{a.s.}} 0$, we have:

$$\frac{\psi(\delta_j^\kappa)}{\delta_j^\kappa} \frac{1}{N} y_j^* \left(\frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} f_i^\kappa v(\delta_i^\kappa) y_i y_i^* \right)^{-1} y_j - \frac{\psi(\delta_j^\kappa)}{\delta_j^\kappa} \frac{1}{N} y_j^* \left(\frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} v(\delta_i^\kappa) y_i y_i^* \right)^{-1} y_j \xrightarrow{\text{a.s.}} 0.$$

As a consequence, for κ sufficiently small and n large enough:

$$\max_{j \in \mathcal{C}_\kappa^c} \left| \frac{\psi(\delta_j^\kappa) q_j}{\delta_j^\kappa} - \psi(\delta_j^\kappa) \right| \leq \kappa',$$

where $\lim_{\kappa \rightarrow 0} \kappa' = 0$. Now, write f_j^κ as:

$$f_j = \frac{\psi \left(\frac{\delta_j^\kappa}{\frac{\psi(\delta_j^\kappa)}{\delta_j^\kappa} q_j} \right)}{\frac{\psi(\delta_j^\kappa)}{\delta_j^\kappa} q_j}.$$

Then, one can easily note that:

$$\lim_{\kappa \rightarrow 0} \limsup_n \max_{j \in \mathcal{C}_\kappa^c} \left\{ |f_j^\kappa - 1| \right\} \rightarrow 0.$$

Combining the results for $j \in \mathcal{C}_\kappa$ and $j \in \mathcal{C}_\kappa^c$, we conclude that for each $\ell > 0$, there exists $\kappa > 0$ small enough such that a.s.,

$$(1 - \ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_i^\kappa)}{\delta_i^\kappa} y_i y_i^* \preceq \frac{1}{n} \sum_{i=1}^n v(q_i) y_i y_i^* \preceq (1 + \ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_i^\kappa)}{\delta_i^\kappa} y_i y_i^*.$$

It remains thus to show that for each $\varepsilon > 0$, there exists κ_0 such that for any $\kappa \leq \kappa_0$ and all large n ,

$$\max_j \left| 1 - \frac{\delta_j}{\delta_j^\kappa} \right| \leq \varepsilon.$$

Recall that $(\delta_1^\kappa, \dots, \delta_n^\kappa)$ are given by:

$$(\delta_1^\kappa, \dots, \delta_n^\kappa) = \lim_{t \rightarrow \infty} (\delta_1^\kappa(t), \dots, \delta_n^\kappa(t))$$

with $\delta_1^\kappa(0), \dots, \delta_n^\kappa(0)$ are arbitrary and:

$$\delta_j^\kappa(t+1) = \frac{1}{N} \text{Tr}(B_N + \tau_j I_N) \left(\frac{1}{n} \sum_{\tau_i \leq M_\kappa} \frac{\phi \circ g_N^{-1}(\delta_i^\kappa(t))}{\delta_i^\kappa(t)} (B_N + \tau_i I_N) \right)^{-1},$$

where we used the relation $\frac{\psi_N}{1+c_N \psi_N} = \phi \circ g_N^{-1}$. Set for $t = 0$, $\delta_i^\kappa = \delta_i$, $i = 1, \dots, n$. We will prove by induction on t that $\delta_j \leq \delta_j^\kappa(t)$ for all $j = 1, \dots, n$, thereby showing that $\delta \leq \delta_j^\kappa$. Obviously, the desired result holds for $t = 0$. Assume now that for all $t \leq k$, $\delta_j^\kappa(t) \geq \delta_j$, and let us show that $\delta_j^\kappa(k+1) \geq \delta_j$. Since $x \mapsto \frac{\phi \circ g_N^{-1}(x)}{x}$ is non-increasing and $\delta_i^\kappa(k) \geq \delta_i$, we have:

$$\frac{\phi \circ g_N^{-1}(\delta_i^\kappa(k))}{\delta_i^\kappa(k)} \leq \frac{\phi \circ g_N^{-1}(\delta_i)}{\delta_i}.$$

Hence,

$$\begin{aligned}\delta_j^\kappa(k+1) &= \frac{1}{N} \operatorname{Tr}(B_N + \tau_j I_N) \left(\frac{1}{n} \sum_{\tau_i \leq M_\kappa} \frac{\phi \circ g_N^{-1}(\delta_i^\kappa(k))}{\delta_i^\kappa(k)} (B_N + \tau_i I_N) \right)^{-1} \\ &\geq \frac{1}{N} \operatorname{Tr}(B_N + \tau_j I_N) \left(\frac{1}{n} \sum_{i=1}^n \frac{\phi \circ g_N^{-1}(\delta_i)}{\delta_i} (B_N + \tau_i I_N) \right)^{-1} \\ &= \delta_j.\end{aligned}$$

We are now in position to control the convergence of $\max_{1 \leq j \leq n} \left| 1 - \frac{\delta_j}{\delta_j^\kappa} \right|$ as $\kappa \rightarrow 0$. In particular, we recall that we need to prove that for each $\varepsilon > 0$, there exists κ_0 such that:

$$\max_j \left| 1 - \frac{\delta_j}{\delta_j^\kappa} \right| \leq \varepsilon.$$

To this end, define the maps T_N, T_N^M as:

$$T_N : (x_1, \dots, x_n) \mapsto \left(\frac{1}{n} \sum_{i=1}^n \frac{\phi \circ g_N^{-1}(x_i)}{x_i} (B_N + \tau_i I_N) \right)^{-1}$$

and

$$T_N^M : (x_1, \dots, x_n) \mapsto \left(\frac{1}{n} \sum_{\substack{i=1 \\ \tau_i \leq M}}^n \frac{\phi \circ g_N^{-1}(x_i)}{x_i} (B_N + \tau_i I_N) \right)^{-1}.$$

From Lemma 4.7 and 4.6, it is easy to see that the spectral norms of $T_N(\delta_1, \dots, \delta_n)$ and $T_N^{M_\kappa}(\delta_1^\kappa, \dots, \delta_n^\kappa)$ are uniformly bounded. Note that:

$$\left\| T_N(\delta_1, \dots, \delta_n) - T_N^M(\delta_1, \dots, \delta_n) \right\| \leq \left\| T_N^M(\delta_1, \dots, \delta_n) \right\|^2 \phi_\infty \frac{\limsup \|B_N\|}{\liminf \delta_N^- \liminf \eta_N^-} \nu_n(M, \infty).$$

and

$$\left\| T_N^{M_\kappa}(\delta_1^\kappa, \dots, \delta_n^\kappa) - T_N^M(\delta_1^\kappa, \dots, \delta_n^\kappa) \right\| \leq \left\| T_N^M(\delta_1^\kappa, \dots, \delta_n^\kappa) \right\|^2 \phi_\infty \frac{\limsup \|B_N\|}{\liminf \delta_N^{-, \kappa} \liminf \eta_N^{-, \kappa}} \nu_n(M, M_\kappa)$$

for any $M_\kappa \geq M$. Setting M large enough so that $\liminf \nu_n(m, M) > 0$, we get:

$$\max \left(\left\| T_N^M(\delta_1, \dots, \delta_n) \right\|, \left\| T_N^M(\delta_1^\kappa, \dots, \delta_n^\kappa) \right\| \right) \leq \frac{\max \left(\limsup (m + \eta_N^+) \delta_N^+, \limsup (m + \eta_N^{+, \kappa}) \delta_N^{+, \kappa} \right)}{m \phi \circ g_N^{-1}(m) (1 - \limsup \nu_n(M, \infty))}.$$

Therefore, one can fix M sufficiently large in such a way that:

$$\limsup_N \left\| T_N(\delta_1, \dots, \delta_n) - T_N^M(\delta_1, \dots, \delta_n) \right\| \leq \frac{\varepsilon}{3} \quad (4.36)$$

and

$$\limsup_N \left\| T_N^{M_\kappa}(\delta_1^\kappa, \dots, \delta_n^\kappa) - T_N^M(\delta_1^\kappa, \dots, \delta_n^\kappa) \right\| \leq \frac{\varepsilon}{3}. \quad (4.37)$$

With this value of M at hand, we will now prove that:

$$\lim_{\kappa \rightarrow 0} \limsup_N \left\| T_N^M(\delta_1^\kappa, \dots, \delta_n^\kappa) - T_N^M(\delta_1, \dots, \delta_n) \right\| = 0.$$

To this end, we will work out the differences $\frac{\delta_j^\kappa - \delta_j}{\delta_j^\kappa}$. We have:

$$\begin{aligned}
\frac{\delta_j^\kappa - \delta_j}{\delta_j^\kappa} &= \\
&\frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\delta_j^\kappa} T_N^{M_\kappa}(\delta_1^\kappa, \dots, \delta_n^\kappa) \left[\frac{1}{n} \sum_{\tau_i \leq M_\kappa} \left(\frac{\phi \circ g_N^{-1}(\delta_i)}{\delta_i} - \frac{\phi \circ g_N^{-1}(\delta_i^\kappa)}{\delta_i^\kappa} \right) (B_N + \tau_i I_N) \right] T_N(\delta_1, \dots, \delta_n) \\
&+ \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\delta_j^\kappa} T_N^{M_\kappa}(\delta_1^\kappa, \dots, \delta_n^\kappa) \left[\frac{1}{n} \sum_{\tau_i \geq M_\kappa} \frac{\phi \circ g_N^{-1}(\delta_i)}{\delta_i} \right] T_N(\delta_1, \dots, \delta_n) \\
&= \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\delta_j^\kappa} T_N^{M_\kappa}(\delta_1^\kappa, \dots, \delta_n^\kappa) \left[\frac{1}{n} \sum_{\tau_i \leq M_\kappa} \left(\frac{\phi \circ g_N^{-1}(\delta_i)}{\delta_i} - \frac{\phi \circ g_N^{-1}(\delta_i^\kappa)}{\delta_i^\kappa} \right) (B_N + \tau_i I_N) \right] T_N(\delta_1, \dots, \delta_n) \\
&+ \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\delta_j^\kappa} T_N^{M_\kappa}(\delta_1^\kappa, \dots, \delta_n^\kappa) \left[\frac{1}{n} \sum_{\tau_i \leq M_\kappa} \phi \circ g_N^{-1}(\delta_i) \left[\frac{1}{\delta_i} - \frac{1}{\delta_i^\kappa} \right] (B_N + \tau_i I_N) \right] T_N(\delta_1, \dots, \delta_n) \\
&+ \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\delta_j^\kappa} T_N^{M_\kappa}(\delta_1^\kappa, \dots, \delta_n^\kappa) \left[\frac{1}{n} \sum_{\tau_i \geq M_\kappa} \frac{\phi \circ g_N^{-1}(\delta_i)}{\delta_i} (B_N + \tau_i I_N) \right] T_N(\delta_1, \dots, \delta_n) \\
&\triangleq \alpha_{1,j} + \alpha_{2,j} + \alpha_{3,j}.
\end{aligned}$$

Note that α_2 can be bounded as:

$$\alpha_{2,j} \leq \max_i \left| \frac{\delta_i^\kappa - \delta_i}{\delta_i^\kappa} \right|.$$

Let j_0 be the index of the maximum element of $\frac{\delta_i^\kappa - \delta_i}{\delta_i^\kappa}, i = 1, \dots, n$. Therefore:

$$-\alpha_{1,j_0} \leq \alpha_{3,j_0}$$

or equivalently,

$$\begin{aligned}
&\frac{1}{N} \text{Tr} \frac{B_N + \tau_{j_0} I_N}{\delta_{j_0}^\kappa} T_N^{M_\kappa}(\delta_1^\kappa, \dots, \delta_n^\kappa) \left[\frac{1}{n} \sum_{\tau_i \leq M_\kappa} \left(\frac{\phi \circ g_N^{-1}(\delta_i^\kappa)}{\delta_i^\kappa} - \frac{\phi \circ g_N^{-1}(\delta_i)}{\delta_i} \right) (B_N + \tau_i I_N) \right] T_N(\delta_1, \dots, \delta_n) \\
&\leq \frac{1}{N} \text{Tr} \frac{B_N + \tau_{j_0} I_N}{\delta_{j_0}^\kappa} T_N^{M_\kappa}(\delta_1^\kappa, \dots, \delta_n^\kappa) \left[\frac{1}{n} \sum_{\tau_i \geq M_\kappa} \frac{\phi \circ g_N^{-1}(\delta_i)}{\delta_i} (B_N + \tau_i I_N) \right] T_N(\delta_1, \dots, \delta_n)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{N} \text{Tr} \frac{B_N + \tau_{j_0} I_N}{\delta_{j_0}^\kappa} T_N^{M_\kappa}(\delta_1^\kappa, \dots, \delta_n^\kappa) \left[\frac{1}{n} \sum_{\tau_i \leq M} \left(\frac{\phi \circ g_N^{-1}(\delta_i^\kappa)}{\delta_i^\kappa} - \frac{\phi \circ g_N^{-1}(\delta_i)}{\delta_i} \right) (B_N + \tau_i I_N) \right] T_N(\delta_1, \dots, \delta_n) \\
&\leq \|T_N(\delta_1, \dots, \delta_n)\| \frac{\phi_\infty \nu_n(M_\kappa, \infty) \limsup \|B_N\|}{\liminf \eta_N^- \delta_N^-}
\end{aligned}$$

The right-hand side in the above inequality converges to zero as $\kappa \rightarrow 0$. This is possible if and only if:

$$\frac{1}{n} \sum_{\tau_i \leq M} \left(\frac{\phi \circ g_N^{-1}(\delta_i^\kappa) - \phi \circ g_N^{-1}(\delta_i)}{\delta_i^\kappa} \right) (B_N + \tau_i I_N) \xrightarrow{\kappa \rightarrow 0} 0. \quad (4.38)$$

Function $x \mapsto \phi \circ g_N^{-1}(x)$ is continuously differentiable. Therefore, by the mean value theorem,

$$\frac{\phi \circ g_N^{-1}(\delta_i^\kappa) - \phi \circ g_N^{-1}(\delta_i)}{\delta_i^\kappa} = \left(\phi \circ g_N^{-1} \right)'(\xi_i^\kappa) \frac{\delta_i^\kappa - \delta_i}{\delta_i^\kappa},$$

where $\left(\phi \circ g_N^{-1} \right)'$ denotes the derivative of $\phi \circ g_N^{-1}$ and $\xi_i^\kappa \in [\delta_i, \delta_i^\kappa]$. Now, since for n large enough, $\min_{i, \tau_i \leq M} \delta_i \geq a \triangleq \liminf \delta_N^- \eta_N^-$ and $\max_{i, \tau_i \leq M} \delta_i^\kappa \leq b \triangleq \limsup \delta_N^{+, \kappa} (\eta_N^{+, \kappa} + M)$, we obtain:

$$\frac{\phi \circ g_N^{-1}(\delta_i^\kappa) - \phi \circ g_N^{-1}(\delta_i)}{\delta_i^\kappa} \geq \inf_{x \in [a, b]} \left(\phi \circ g_N^{-1} \right)'(x) \frac{\delta_i^\kappa - \delta_i}{\delta_i^\kappa}$$

Since $\inf_{x \in [a, b]} \left(\phi \circ g_N^{-1} \right)' > 0$ by Assumption 2-ii), we get:

$$\frac{1}{n} \sum_{\tau_i \leq M} \frac{\delta_i^\kappa - \delta_i}{\delta_i^\kappa} (B_N + \tau_i I_N) \xrightarrow{\kappa \rightarrow 0} 0. \quad (4.39)$$

Using the convergences (4.38) and (4.39), we can prove that:

$$\lim_{\kappa \rightarrow 0} \limsup_N \left\| T_N^M(\delta_1^\kappa, \dots, \delta_n^\kappa) - T_N^M(\delta_1, \dots, \delta_n) \right\| \rightarrow 0.$$

This can be easily seen by noting that:

$$\begin{aligned} & T_N^M(\delta_1, \dots, \delta_n) - T_N^M(\delta_1^\kappa, \dots, \delta_n^\kappa) \\ &= T_N^M(\delta_1, \dots, \delta_n) \frac{1}{n} \sum_{\tau_i \leq M} \left\{ \frac{\phi \circ g_N^{-1}(\delta_i^\kappa) - \phi \circ g_N^{-1}(\delta_i)}{\delta_i^\kappa} + \phi \circ g_N^{-1}(\delta_i) \left[\frac{1}{\delta_i^\kappa} - \frac{1}{\delta_i} \right] \right\} (B_N + \tau_i I_N) \\ &\times T_N^M(\delta_1^\kappa, \dots, \delta_n^\kappa). \end{aligned}$$

One can thus choose κ_0 in such a way that for all $\kappa \leq \kappa_0$

$$\limsup \left\| T_N^M(\delta_1^\kappa, \dots, \delta_n^\kappa) - T_N^M(\delta_1, \dots, \delta_n) \right\| \leq \frac{\varepsilon}{3}.$$

From (4.36) and (4.37), we therefore get for all $\kappa \leq \kappa_0$

$$\limsup_N \left\| T_N(\delta_1, \dots, \delta_n) - T_N^{M, \kappa}(\delta_1^\kappa, \dots, \delta_n^\kappa) \right\| \leq \varepsilon.$$

In an equivalent way, we therefore have, for each $\varepsilon > 0$, there exists κ_0 such that for any $\kappa \leq \kappa_0$ and all large n ,

$$\max_{1 \leq j \leq n} \left| 1 - \frac{\delta_j}{\delta_j^\kappa} \right| \leq \varepsilon.$$

Using this result, we will show that for each $\ell > 0$, there exist $\kappa > 0$ small enough, such that a.s.,

$$(1 - \ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi_N(\delta_i)}{\delta_i} y_i y_i^* \preceq \frac{1}{n} \sum_{i=1}^n \frac{\psi_N(\delta_i^\kappa)}{\delta_i^\kappa} y_i y_i^* \preceq (1 + \ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi_N(\delta_i)}{\delta_i} y_i y_i^*.$$

To this end, it suffices to show that for each $\varepsilon > 0$ and κ small enough:

$$\max_{1 \leq i \leq n} |\psi_N(\delta_i) - \psi_N(\delta_i^\kappa)| \leq \varepsilon.$$

If this was not true, then one can find a sequence (n) over which:

$$\max_{1 \leq i \leq (n)} \left| \psi_{N(n)}(\delta_i) - \psi_{N(n)}(\delta_i^\kappa) \right| \geq \epsilon. \quad (4.40)$$

for any small κ . Since the sequence function ψ_N converge uniformly, one can extract a subsequence (p) from (n) such that $c_{(p)} \rightarrow c^*$ and $\psi_{(p)}$ converge uniformly to ψ^* . On the other hand, we know that for any arbitrairly small r there exists κ_0 such that for any $\kappa \leq \kappa_0$ and for all large n ,

$$\max_{1 \leq j \leq n} \left| 1 - \frac{\delta_j}{\delta_j^\kappa} \right| \leq r.$$

or also, for all $j = 1, \dots, n$,

$$\delta_j \geq (1 - r)\delta_j^\kappa.$$

Let x_0 be such that $\psi^*(x_0(1 - r)) \geq \psi_\infty(1 - \epsilon/3)$. Since ψ^* is increasing and bounded at infinity by ψ_∞ , we have, for any $x, y \geq x_0(1 - r)$,

$$|\psi^*(x) - \psi^*(y)| \leq \frac{\epsilon}{3}.$$

Consider the indices i such that $\delta_i^\kappa \geq x_0$, and thus $\delta_i \geq x_0(1 - r)$. Take n large enough such that:

$$\|\psi_N - \psi^*\| \leq \frac{\epsilon}{3}.$$

Then, for those indices, one can prove that:

$$\begin{aligned} \max_{\substack{1 \leq j \leq (n) \\ \delta_j^\kappa \geq x_0}} \left| \psi_{N(n)}(\delta_j) - \psi_{N(n)}(\delta_j^\kappa) \right| &\leq \max_{\substack{1 \leq j \leq (n) \\ \delta_j^\kappa \geq x_0}} \left| \psi_{N(n)}(\delta_j) - \psi^*(\delta_j) \right| \\ &\quad + \left| \psi^*(\delta_j) - \psi^*(\delta_j^\kappa) \right| + \left| \psi^*(\delta_j^\kappa) - \psi_{N(n)}(\delta_j^\kappa) \right| \leq \epsilon. \end{aligned}$$

Consider now the indices i such that $\delta_i^\kappa \leq x_0$. For those indices, we have:

$$\begin{aligned} \psi(\delta_i^\kappa) - \psi(\delta_i)u(0) &\leq \delta_i^\kappa - \delta_i \\ &= u(0) \frac{\delta_i^\kappa - \delta_i}{x_0} x_0 \\ &\leq u(0)x_0 \frac{\delta_i^\kappa - \delta_i}{\delta_i^\kappa}. \end{aligned}$$

Taking $r \leq \frac{\epsilon}{x_0 u(0)}$, we will get:

$$|\psi(\delta_i^\kappa) - \psi(\delta_i)| \leq \epsilon$$

which is in contradiction with (4.40). We therefore have for each $\ell > 0$,

$$(1 - \ell)^2 \frac{1}{n} \sum_{i=1}^n \frac{\psi_N(\delta_i)}{\delta_i} y_i y_i^* \preceq \hat{C}_N \preceq (1 + \ell)^2 \frac{1}{n} \sum_{i=1}^n \frac{\psi_N(\delta_i)}{\delta_i} y_i y_i^*$$

which therefore implies that $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$. This completes the proof.

APPENDIX: TECHNICAL LEMMAS

This appendix gathers some technical Lemmas that will help control quadratic forms of the type:

$$z_j^* R_j \left(\frac{1}{n} \sum_{i=1, i \neq j}^n R_i z_i z_i^* R_i^* \right)^{-1} R_j z_j,$$

where z_1, \dots, z_n are independent random vectors with size $\bar{N} \times 1$ and R_1, \dots, R_n are n matrices of size $\bar{N} \times \bar{N}$ independent of z_1, \dots, z_n and whose eigenvalues are bounded above and below by constants independent of n and N . The control of this quadratic form can be performed using the most well-known trace Lemma of Silverstein et al [2, Lemma 2.7], provided that we can guarantee that the infimum of the set \mathcal{S} of smallest eigenvalues of matrices $\frac{1}{n} \sum_{i=1, i \neq j}^n R_i z_i z_i^* R_i, j = 1, \dots, n$ is above zero uniformly in n and N or more formally,

$$\min_{1 \leq j \leq n} \left\{ \lambda_1 \left(\frac{1}{n} \sum_{i=1, i \neq j}^n R_i z_i z_i^* R_i^* \right), j = 1, \dots, n \right\} > \epsilon,$$

for some $\epsilon > 0$ a.s. for n large enough. This is however a challenging task since the question of the smallest eigenvalue of matrices of the form $\frac{1}{n} \sum_{i=1}^n R_i z_i z_i^* R_i^*$ being above zero almost surely was implicitly raised in [19] where this fact was assumed because no immediate answer can be provided in general. It was only recently that we have provided a rigorous proof thereof under the Gaussian setting [1]. In this appendix, we extend this result to the random vector model of the present work, i.e. $z_i = [s_i, w_i]$ with $s_i \sim \mathcal{CN}(0, I_K)$ and w_i zero-mean unitarily invariant satisfying $\|w_i\|^2 = N$. The control of the infimum of the set \mathcal{S} will be shown along the same lines of the proof of Lemma 1 in [7].

In the sequel, we will start by bounding the maximum eigenvalue of $\frac{1}{n} \sum_{i=1}^n R_i z_i z_i^* R_i^*$ when z_1, \dots, z_n are Gaussian random vectors. To this end, we will start by introducing the following concentration inequality, the proof of which is provided for sake of completeness.

Lemma .11. *Let $\gamma_1, \dots, \gamma_n$ be n independent random variables having an exponential distribution with rate parameter 1, and $(\alpha_i)_{i=1}^n$ be positive scalars. Then, there exists C such that for any $t > 0$:*

$$\mathbb{P} \left[\sum_{i=1}^n \alpha_i \gamma_i > t \right] \leq C \exp \left(- \min \left(\frac{t^2}{4 \sum_{i=1}^n \alpha_i^2}, \frac{t}{4 \max_{1 \leq i \leq n} \alpha_i} \right) \right).$$

Proof. Let s be a positive scalar such that $s < \frac{1}{2\alpha_i}$ for all $i = 1, \dots, n$. Then:

$$\begin{aligned} \mathbb{P} \left[\sum_{i=1}^n \alpha_i \gamma_i > t \right] &= \mathbb{P} \left[\exp \left(s \sum_{i=1}^n \alpha_i \gamma_i \right) > \exp(ts) \right] \\ &\leq \frac{\exp(-ts)}{\prod_{j=1}^n (1 - s\alpha_j)}. \end{aligned}$$

Now, using the inequality $-\log(1-x) \leq x + x^2$ for $x \leq \frac{1}{2}$, we get:

$$\frac{1}{1 - s\alpha_j} \leq \exp(s\alpha_j) \exp(s^2\alpha_j^2) \leq \exp\left(\frac{1}{2}\right) \exp(s^2\alpha_j^2),$$

thereby yielding:

$$\mathbb{P} \left[\sum_{i=1}^n \alpha_i \gamma_i > t \right] \leq \exp\left(\frac{1}{2}\right) \exp\left(-ts + \sum_{i=1}^n s^2 \alpha_i^2\right).$$

Two cases have to be considered. If $\frac{t \max_{1 \leq i \leq n} \alpha_i}{2 \sum_{i=1}^n \alpha_i^2} \leq \frac{1}{2}$. Then, setting $s = \frac{t}{2 \sum_{i=1}^n \alpha_i^2}$ yields:

$$\mathbb{P} \left[\sum_{i=1}^n \alpha_i \gamma_i > t \right] \leq C \exp \left(-\frac{t^2}{4 \sum_{i=1}^n \alpha_i^2} \right) \quad (.41)$$

where $C = \exp(\frac{1}{2})$. Otherwise, if $\frac{t \max_{1 \leq i \leq n} \alpha_i}{2 \sum_{i=1}^n \alpha_i^2} \geq \frac{1}{2}$. Then, $\sum_{i=1}^n \alpha_i^2 \leq t \max_{1 \leq i \leq n} \alpha_i$. Set $s = \frac{1}{2 \max_{1 \leq i \leq n} \alpha_i}$, then we have:

$$\mathbb{P} \left[\sum_{i=1}^n \alpha_i \gamma_i > t \right] \leq C \exp \left(-\frac{t}{4 \max_{1 \leq i \leq n} \alpha_i} \right) \quad (.42)$$

Gathering (.41) and (.42) yields the desired result. \square

With this Lemma at hand, we will now control the maximum eigenvalue of $\frac{1}{n} R_i z_i z_i^* R_i$. We have in particular, the following result:

Lemma .12. *Let z_1, \dots, z_n be n independent $\overline{N} \times 1$ standard Gaussian vectors. Consider $(R_i)_{i=1}^n$ a family of $N \times \overline{N}$ matrices with uniformly bounded spectral norm, i.e.,*

$$\limsup_n \max_{1 \leq i \leq n} \|R_i\| < +\infty.$$

Let Σ be given by:

$$\Sigma = \frac{1}{n} \sum_{i=1}^n R_i z_i z_i^* R_i^*$$

Then, there exists a constant K_{\max} such that a.s. for n large enough,

$$\|\Sigma\| < K_{\max}.$$

Proof. The proof relies on the observation that:

$$\|\Sigma\| = \max_{\|a\|=1} a^* \Sigma a.$$

Based on the result of Lemma .11, a concentration inequality involving the term $a^* \Sigma a$ can be established. Define $u_i = \frac{R_i a}{\|R_i a\|}$ and expand $a^* \Sigma a$ as:

$$\begin{aligned} a^* \Sigma a &= \frac{1}{n} \sum_{i=1}^n a^* R_i z_i z_i^* R_i a \\ &= \frac{1}{n} \sum_{i=1}^n (a^* R_i R_i^* a) |u_i^* z_i|^2. \end{aligned}$$

Since u_i is unitary, the random quantity $u_i^* z_i$ is a Gaussian random variable with zero mean and variance 1. Hence, $|z_i^* u_i|^2, i = 1, \dots, n$ is a sequence of n independent exponential

distributed random variables with rate parameter 1. Applying Lemma .11, we get:

$$\begin{aligned} \mathbb{P}[a^* \Sigma a > t] &= \mathbb{P}\left[\sum_{i=1}^n (a^* R_i R_i^* a) |u_i^* z_i|^2 > nt\right] \\ &\leq C \exp\left(-\min\left(\frac{n^2 t^2}{4 \sum_{i=1}^n a^* R_i R_i^* a}, \frac{nt}{4 \max_{1 \leq i \leq n} a^* R_i R_i^* a}\right)\right), \end{aligned}$$

where C is some constant independent of n and N . For $t \geq 1$, we therefore have:

$$\mathbb{P}[a^* \Sigma a > t] \leq C \exp\left(-\frac{nt}{4 \max_{1 \leq i \leq n} \|R_i\|^2}\right). \quad (.43)$$

With the above inequality at hand, we are now in position to control the behaviour of the spectral norm of Σ . For that, we will resort to the well-known ϵ -net argument. Let \mathcal{S} be an $\frac{1}{2}$ -net of the unit sphere of \mathbb{C}^N . Using Lemma 2.3.2 of [16], we have:

$$\mathbb{P}\left[\left\|\Sigma^{\frac{1}{2}}\right\| \geq t\right] \leq \mathbb{P}\left[\bigcup_{a \in \mathcal{S}} a^* \Sigma a > \frac{t^2}{4}\right],$$

Using (.43), we obtain that each of the probabilities of $\mathbb{P}\left[a^* \Sigma a > \frac{t^2}{4}\right]$ is bounded by $C \exp(-\frac{ct^2 n}{4})$ for t and n large enough with c some constant independent of n . On the other hand, the cardinality of \mathcal{S} is of order $(\mathcal{O}(1))^n$. By taking t large enough, this term can be absorbed into the exponential gain of $C \exp(-\frac{cnt}{4})$. For some t large enough the event $\left\{\left\|\Sigma^{\frac{1}{2}}\right\| < t\right\}$ holds with overwhelming probability. Setting $K_{\max} \geq t$. We have thus a.s. for n large enough,

$$\left\|\Sigma^{\frac{1}{2}}\right\| < K_{\max}.$$

□

All the above results are derived under the assumption of a Gaussian setting. Before going further into the proofs of the main lemmas of this appendix, we will show that the considered random model of the paper is equivalent to a Gaussian model. In particular, we have the following result:

Lemma .13. *Let $z_1, \dots, z_n \in \mathbb{C}^{\overline{N}}$ be n independent and identically distributed vectors such that $z_i = [s_i^T, w_i^T]^T$ where s_i and w_i are independent and distributed as:*

- $s_i \sim \mathcal{CN}(0, I_K)$
- w_i is unitarily invariant zero-mean vector such that $\|w_i\| = N$.

Write w_i as $w_i = \frac{\sqrt{N} \tilde{w}_i}{\|\tilde{w}_i\|}$ with $\tilde{w}_i \sim \mathcal{CN}(0, I_N)$, and denote by Σ the $N \times N$ matrix given by:

$$\Sigma = \frac{1}{n} \sum_{i=1}^n R_i z_i z_i^* R_i^*$$

where $R_i = [R_{i,1} \ R_{i,2}]$, $R_{i,1} \in \mathbb{C}^{N \times K}$ and $R_{i,2} \in \mathbb{C}^{N \times N}$ are some deterministic matrices with uniformly bounded norm. Let $\tilde{z}_i = [s_i^T, \tilde{w}_i^T]^T$, and $\tilde{\Sigma}$ be given by:

$$\tilde{\Sigma} = \frac{1}{n} \sum_{i=1}^n R_i \tilde{z}_i \tilde{z}_i^* R_i^*$$

Then, in the asymptotic regime,

$$\|\Sigma - \tilde{\Sigma}\| \xrightarrow{a.s.} 0.$$

Proof. Notice that Σ can be written as:

$$\Sigma = \frac{1}{n} \sum_{i=1}^n R_i D_i \tilde{z}_i \tilde{z}_i^* D_i R_i^*$$

where

$$D_i = \begin{bmatrix} I_K & 0 \\ 0 & \frac{\sqrt{N}}{\|\tilde{w}_i\|} I_N \end{bmatrix}.$$

Then,

$$\begin{aligned} \Sigma - \tilde{\Sigma} &= \frac{1}{n} \sum_{i=1}^n R_i (D_i - I) \tilde{z}_i \tilde{z}_i^* (D_i - I) R_i^* + \frac{1}{n} \sum_{i=1}^n R_i \tilde{z}_i \tilde{z}_i^* (D_i - I) R_i^* \\ &\quad + \frac{1}{n} \sum_{i=1}^n R_i (D_i - I) \tilde{z}_i \tilde{z}_i^* R_i^* \\ &= \Theta_1 + \Theta_2 + \Theta_3. \end{aligned}$$

In the sequel, we will prove that the spectral norms of $\Theta_i, i = 1, 2, 3$ converge to zero almost surely. We will treat only the term Θ_2 since the treatment of Θ_1 and Θ_3 relies on the same arguments. Expanding Θ_2 using $R_i = [R_{i,1} \ R_{i,2}]$, we get:

$$\begin{aligned} \Theta_2 &= \frac{1}{n} \sum_{i=1}^n R_{i,1} s_i \tilde{w}_i^* \left(\frac{\sqrt{N}}{\|\tilde{w}_i\|} - 1 \right) R_{i,2}^* + \frac{1}{n} \sum_{i=1}^n R_{i,2} \tilde{w}_i \tilde{w}_i^* R_{i,2}^* \left(\frac{\sqrt{N}}{\|\tilde{w}_i\|} - 1 \right) \\ &\triangleq \Theta_{1,2} + \Theta_{2,2}. \end{aligned}$$

Let us control $\Theta_{1,2}$.

$$\begin{aligned} \|\Theta_{1,2}\| &= \sup_{\|a\|=1, \|b\|=1} \frac{1}{n} \left| \sum_{i=1}^n a^* R_{i,1} s_i \tilde{w}_i^* R_{i,2}^* b \left(\frac{\sqrt{N}}{\|\tilde{w}_i\|} - 1 \right) \right| \\ &\leq \max_i \left| \frac{\sqrt{N}}{\|\tilde{w}_i\|} - 1 \right| \sup_{\|a\|=1, \|b\|=1} \frac{1}{n} \sum_{i=1}^n |a^* R_{i,1} s_i| |\tilde{w}_i^* R_{i,2}^* b| \\ &\leq \max_i \left| \frac{\sqrt{N}}{\|\tilde{w}_i\|} - 1 \right| \sqrt{\sup_{\|a\|=1} \frac{1}{n} \sum_{i=1}^n a^* R_{i,1} s_i s_i^* R_{i,1}^* a} \sqrt{\sup_{\|b\|=1} \frac{1}{n} \sum_{i=1}^n \tilde{w}_i^* R_{i,2} b b^* R_{i,2}^* \tilde{w}_i} \\ &\leq \max_i \left| \frac{\sqrt{N}}{\|\tilde{w}_i\|} - 1 \right| \sqrt{\left\| \frac{1}{n} \sum_{i=1}^n R_{i,1} s_i s_i^* R_{i,1}^* \right\|} \sqrt{\left\| \frac{1}{n} \sum_{i=1}^n R_{i,2} \tilde{w}_i \tilde{w}_i^* R_{i,2}^* \right\|} \end{aligned}$$

Using the facts that $\max_i \left| \frac{\sqrt{N}}{\|\tilde{w}_i\|} - 1 \right|$ converges almost surely to zero and $\left\| \frac{1}{n} \sum_{i=1}^n R_{i,1} s_i s_i^* R_{i,1} \right\|$ and $\left\| \frac{1}{n} \sum_{i=1}^n R_{i,2} \tilde{w}_i \tilde{w}_i^* R_{i,2} \right\|$ are almost surely bounded as a result of Lemma .12, we have:

$$\|\Theta_{1,2}\| \xrightarrow{\text{a.s.}} 0.$$

Similarly, we can also prove that:

$$\|\Theta_{2,2}\| \xrightarrow{\text{a.s.}} 0,$$

thereby implying that:

$$\|\Theta_2\| \xrightarrow{\text{a.s.}} 0.$$

□

Using the same notations of Lemma .13, consider Σ_j and $\tilde{\Sigma}_j$ the $N \times N$ matrices given by:

$$\begin{aligned} \Sigma_j &= \frac{1}{n} \sum_{i=1, i \neq j}^n R_i z_i z_i^* R_i \\ \tilde{\Sigma}_j &= \frac{1}{n} \sum_{i=1, i \neq j}^n R_i \tilde{z}_i \tilde{z}_i^* R_i. \end{aligned}$$

Arguing along the same lines as in the proof of Lemma .13, we can show that:

$$\max_{1 \leq j \leq n} \left\| \Sigma_j - \tilde{\Sigma}_j \right\| \xrightarrow{\text{a.s.}} 0. \quad (.44)$$

This observation is essential to facilitate the proof of the first main result of this Appendix which is about showing that:

$$\min_{1 \leq j \leq n} \left\{ \lambda_1(\Sigma_j), j = 1, \dots, n \right\} > \epsilon$$

In fact, from the convergence inequality in (.44), we can see that the proof can be reduced to showing this result when Σ_j is replaced with $\tilde{\Sigma}_j$. The proof of the following Lemma will rely on this observation:

Lemma .14. *Let $z_1, \dots, z_n \in \mathbb{C}^{\bar{N}}$ be n independent and identically distributed vectors such that $z_i = [s_i^T, w_i^T]^T$ where s_i and w_i are independent and distributed as:*

- $s_i \sim \mathcal{CN}(0, I_K)$
- w_i is unitarily invariant zero-mean vector such that $\|w_i\| = N$.

Define matrices Σ and Σ_j as:

$$\begin{aligned} \Sigma &= \frac{1}{n} \sum_{i=1}^n R_i z_i z_i^* R_i \\ \Sigma_j &= \frac{1}{n} \sum_{i=1, i \neq j}^n R_i z_i z_i^* R_i, \end{aligned}$$

where $(R_i)_{i=1}^n$ are $N \times \bar{N}$ matrices satisfying:

$$\liminf_N \min_{1 \leq i \leq n} \lambda_1(R_i R_i^*) > 0$$

and

$$\limsup_N \max_{1 \leq i \leq n} \lambda_N(R_i R_i^*) < +\infty,$$

Consider the asymptotic regime of Assumption 1. Therefore, there exists $\epsilon > 0$ such that for all large n a.s.,

$$\lambda_1(\Sigma) \geq \min_{1 \leq j \leq n} \lambda_1(\Sigma_j) > \epsilon.$$

Proof. It is clear from the discussion before the statement of the above lemma that we can assume z_1, \dots, z_n to be standard Gaussian vectors. Under the Gaussian setting, the fact that the smallest eigenvalue of Σ is almost surely bounded above zero can be deduced from corollary 5 of our work in [1]. It remains thus to treat that of Σ_j . To this end, we will resort to the same kind of the arguments as those used in the proof of [7, Lemma 1]. Notice first that we can assume without loss of generality that $\lambda_1(\Sigma_j) \neq \lambda_1(\Sigma)$, for all $j = 1, \dots, n$. By definition, the eigenvalues of Σ_j are solutions in λ of the following equation:

$$\det(\Sigma_j - \lambda I_N) = 0.$$

Developing the above equation, we obtain:

$$\begin{aligned} \det(\Sigma_j - \lambda I_N) &= \det\left(\Sigma - \frac{1}{n} R_j z_j z_j^* R_j^* - \lambda I_N\right) \\ &= \det Q(\lambda) \det\left(I_N - Q(\lambda)^{\frac{1}{2}} \frac{1}{n} R_j z_j z_j^* R_j^* Q(\lambda)^{\frac{1}{2}}\right) \\ &= \det Q(\lambda) \left(1 - \frac{1}{n} z_j^* R_j^* Q(\lambda) R_j z_j\right), \end{aligned}$$

where $Q(\lambda) = (\Sigma - \lambda I_N)^{-1}$. If λ is an eigenvalue of Σ_j different from that of Σ , then necessarily:

$$\frac{1}{n} z_j^* R_j^* Q(\lambda) R_j z_j = 1.$$

Building on the ideas of [7], we propose to study the behaviour of function:

$$f_{n,j}(x) = \frac{1}{n} z_j^* R_j^* Q(x) R_j z_j,$$

in a neighborhood of zero. The result of the lemma follows if we prove that there exists $\xi > 0$ such that $f_{n,j}(x) < 1$ for all $x \in [0, \xi]$ and $j = 1, \dots, n$ a.s. for n large enough. From our recent result in [1], we know that there exists $\eta > 0$ such that a.s. for n large enough, $\lambda_1(\Sigma) > \eta$. Functions $x \mapsto f_{n,j}(x)$ being increasing in the interval $[0, \eta]$, it suffices thus to show that there exists ξ in $[0, \eta]$ such that $f_{n,j}(\xi) < 1$ a.s. for n large enough.

Let us start by analyzing the behaviour of $f_{n,j}(x)$ for $x < 0$. Define $Q_j(x)$ as $Q_j(x) = (\Sigma_j - x I_N)^{-1}$. Using the matrix inversion relation: $a^*(A + a a^*)^{-1} a = \frac{a^* A^{-1} a}{1 + a^* A^{-1} a}$ for $a \in \mathbb{C}^{N \times 1}$ and A any $N \times N$ invertible matrix, we obtain:

$$f_{n,j}(x) = \frac{\frac{1}{n} z_j^* R_j^* Q_j(x) R_j z_j}{1 + \frac{1}{n} z_j^* R_j^* Q_j(x) R_j z_j} = 1 - \frac{1}{1 + \frac{1}{n} z_j^* R_j^* Q_j(x) R_j z_j}.$$

Now, for $x < 0$, using the trace lemma of Silverstein et al [2, Lemma 2.7] in conjunction with the rank-one perturbation Lemma [15, Lemma 2.6], we can prove that:

$$\max_{1 \leq j \leq n} \left| \frac{1}{n} z_j^* R_j^* Q_j(x) R_j z_j - \frac{1}{n} \text{Tr} Q(x) R_j R_j^* \right| \xrightarrow{\text{a.s.}} 0.$$

and thus:

$$\max_{1 \leq j \leq n} \left| f_{n,j}(x) - 1 + \frac{1}{1 + \frac{1}{n} \text{Tr} Q(x) R_j R_j^*} \right| \xrightarrow{\text{a.s.}} 0.$$

Therefore, for $\epsilon < \min_{1 \leq j \leq n} \frac{1/2}{1 + \frac{\lambda_N(R_j R_j^*)}{\eta}}$ and $x < 0$, we have for n large enough a.s.,

$$\forall j = 1, \dots, n \quad f_{n,j}(x) \leq 1 - \frac{1}{1 + \frac{1}{n} \text{Tr} Q R_j R_j^*} + \epsilon. \quad (.45)$$

On the other hand, since the smallest eigenvalue of Σ is greater than η , we have for n large enough, a.s.

$$\frac{1}{n} \text{Tr} R_j R_j^* Q \leq \frac{\lambda_N(R_j R_j^*)}{\eta} \quad (.46)$$

Plugging (.46) into (.45), we obtain that for each $x < 0$ we have for n large enough, a.s.

$$\forall j = 1, \dots, n \quad f_{n,j}(x) \leq 1 - \epsilon. \quad (.47)$$

Now, we will consider the analysis of functions $f_{n,j}(x)$ on the open interval $U = (-\frac{\eta}{2}, \frac{\eta}{2})$. Note that on this interval, functions $x \mapsto f_{n,i}(x), i = 1, \dots, n$ are well-defined and continuously differentiable. Moreover, for each $x \in U$, we have:

$$f'_{n,j}(x) = \frac{1}{n} z_j^* R_j^* Q^2(x) R_j z_j \leq \frac{\frac{1}{n} z_j^* R_j^* R_j z_j}{(\lambda_1(\Sigma) - \frac{\eta}{2})^2}.$$

Moreover, we have:

$$\max_{1 \leq j \leq n} \left| \frac{1}{n} z_j^* R_j^* R_j z_j - \frac{1}{n} \text{Tr} R_j R_j^* \right| \xrightarrow{\text{a.s.}} 0,$$

The above convergence along with the fact that $\lambda_1(\Sigma) > \eta$ for n large enough a.s. yields:

$$0 < f'_{n,j}(x) < \frac{2 \limsup_n \max_{1 \leq j \leq n} \frac{1}{n} \text{Tr} R_j R_j^*}{\frac{\eta^2}{4}} \triangleq K.$$

By bounding the derivatives of functions $f_{n,j}$ over U , we have by the mean value theorem:

$$\forall x \in [0, \frac{\eta}{2}], \quad f_{n,j}(x) < f_{n,j}(-x) + 2xK'$$

Set $\xi = \min(\frac{\eta}{2}, \frac{\epsilon}{4K})$. Then, we know from (.47) that:

$$f_{n,j}(-\xi) \leq 1 - \frac{1/2}{1 + \frac{\lambda_N(R_j R_j^*)}{\eta}}$$

Combining this inequality with the fact that $f_{n,j}(\xi) < f_{n,j}(-\xi) + 2\xi K'$, yields:

$$f_{n,j}(\xi) \leq 1 - \frac{\epsilon}{4K},$$

thereby finishing the proof. \square

We are now in position to state the following key results of this appendix:

Lemma .15. *Let $z_1, \dots, z_n \in \mathbb{C}^{\overline{N}}$ be n independent and identically distributed vectors such that $z_i = [s_i^T, w_i^T]^T$ where s_i and w_i are independent and distributed as:*

- $s_i \sim \mathcal{CN}(0, I_K)$
- w_i is unitarily invariant zero-mean vector such that $\|w_i\| = N$.

Let $(A_{N,j})_{j=1}^n$ be random matrices independent of z_1, \dots, z_n and κ be a positive constant. Then,

$$\max_{1 \leq j \leq n} 1_{\|A_j\| \leq \kappa} \left| \frac{1}{N} z_j^* A_{N,j} z_j - \frac{1}{N} \text{Tr} A_{N,j} \right| \xrightarrow{\text{a.s.}} 0.$$

Proof. Write $w_i = \frac{\sqrt{N}\tilde{w}_i}{\|\tilde{w}_i\|}$ with $\tilde{w}_i \in \mathbb{C}^{\overline{N}-K}$. Let D_j be the diagonal matrix given by:

$$D_j = \begin{bmatrix} I_K & 0 \\ 0 & \frac{\sqrt{N}}{\|\tilde{w}_i\|} \end{bmatrix}$$

Let $\tilde{z}_i = [s_i^T, \tilde{w}_i^T]^T$. Then:

$$\begin{aligned} 1_{\|A_j\| \leq \kappa} \frac{1}{N} z_j^* A_{N,j} z_j &= 1_{\|A_j\| \leq \kappa} \frac{1}{N} \tilde{z}_j^* D_j A_{N,j} D_j \tilde{z}_j \\ &= 1_{\|A_j\| \leq \kappa} \frac{1}{N} \tilde{z}_j^* (D_j - I_{\overline{N}}) A_{N,j} (D_j - I_{\overline{N}}) \tilde{z}_j + 1_{\|A_j\| \leq \kappa} \frac{1}{N} \tilde{z}_j^* A_{N,j} (D_j - \overline{N}) \tilde{z}_j \\ &\quad + 1_{\|A_j\| \leq \kappa} \frac{1}{N} \tilde{z}_j^* (D_j - I_{\overline{N}}) A_{N,j} \tilde{z}_j + 1_{\|A_j\| \leq \kappa} \frac{1}{N} \tilde{z}_j^* A_{N,j} \tilde{z}_j. \end{aligned} \quad (.48)$$

Since $\max_j \left| 1 - \frac{\sqrt{N}}{\|\tilde{w}_i\|} \right| \xrightarrow{\text{a.s.}} 0$,

$$\max_{1 \leq j \leq n} \|D_j - I_{\overline{N}}\| \xrightarrow{\text{a.s.}} 0.$$

Therefore, it is easy to see that the maximum over j of the first three terms in (.51) converge to zero almost surely. The problem unfolds thus to the control of $1_{\|A_{N,j}\| \leq \kappa} \frac{1}{N} \tilde{z}_j^* A_{N,j} \tilde{z}_j$. Let $\mathbb{E}_{\tilde{z}_j}$ denote the expectation with respect to \tilde{z}_j . From the trace Lemma of Silverstein et al [2, Lemma 2.7] applied for $p > 2$, we obtain:

$$\mathbb{E}_{\tilde{z}_j} \left[1_{\|A_{N,j}\| \leq \kappa} \left| \frac{1}{N} \tilde{z}_j^* A_{N,j} \tilde{z}_j - \frac{1}{N} \text{Tr } A_{N,j} \right|^p \right] \leq \frac{1_{\|A_j\| \leq \kappa} K_p}{N^{\frac{p}{2}}} \left[\left(\frac{\zeta_4}{N} \text{Tr } A_{N,j}^2 \right)^{\frac{p}{2}} + \frac{\zeta_{2p}}{N^{\frac{p}{2}}} \text{Tr } A_{N,j}^p \right]$$

where ζ is any upper-bound on $\mathbb{E} \left[|\tilde{z}_{i,j}|^\ell \right]$ and K_p a constant dependent only in p . Since $1_{\|A_{N,j}\| \leq \kappa} \|A_{N,j}\| \leq \kappa$, we have:

$$\mathbb{E}_{\tilde{z}_j} \left[1_{\|A_{N,j}\| \leq \kappa} \left| \frac{1}{N} \tilde{z}_j^* A_{N,j} \tilde{z}_j - \frac{1}{N} \text{Tr } A_{N,j} \right| \right] \leq \frac{K_p \kappa^p}{N^{\frac{p}{2}}} \left(\zeta_4^{\frac{p}{2}} + \frac{\zeta_{2p}}{N^{\frac{p}{2}-1}} \right)$$

This bound being independent of $A_{N,j}$, we can take the expectation with respect to $A_{N,j}$ to obtain:

$$\mathbb{E} \left[1_{\|A_{N,j}\| \leq \kappa} \left| \frac{1}{N} \tilde{z}_j^* A_{N,j} \tilde{z}_j - \frac{1}{N} \text{Tr } A_{N,j} \right|^p \right] = \mathcal{O} \left(\frac{1}{N^{\frac{p}{2}}} \right).$$

Therefore,

$$\max_{1 \leq j \leq n} 1_{\|A_{N,j}\| \leq \kappa} \left| \frac{1}{N} \tilde{z}_j^* A_{N,j} \tilde{z}_j - \frac{1}{N} \text{Tr } A_{N,j} \right| \xrightarrow{\text{a.s.}} 0.$$

□

Lemma .16. Let $z_1, \dots, z_n \in \mathbb{C}^{\overline{N}}$ be n independent and identically distributed vectors such that $z_i = [s_i^T, w_i^T]^T$ where $s_i \in \mathbb{C}^K$ and $w_i \in \mathbb{C}^{\overline{N}-K}$ are independent and distributed as:

- $s_i \sim \mathcal{CN}(0, I_K)$
- w_i is unitarily invariant zero-mean vector such that $\|w_i\| = N$ where $N \leq \overline{N}$.

Denote by (R_i) a family of $N \times \overline{N}$ deterministic matrix that satisfy

$$\limsup_n \max_{1 \leq i \leq n} \lambda_N(R_i R_i^*) < +\infty$$

Consider the asymptotic regime of Assumption 1. Let Σ_j be given as:

$$\Sigma_j = \frac{1}{n} \sum_{\substack{i=1 \\ i \neq j}}^n R_i z_i z_i^* R_i^*$$

Assume that there exists $\epsilon > 0$ such that for all large n a.s.,

$$\lambda_1(\Sigma) \geq \min_{1 \leq j \leq n} \lambda_1(\Sigma_j) > \epsilon$$

Then, for any $\Theta_N \in \mathbb{C}^{N \times N}$ with bounded spectral norm,

$$\max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} - \frac{1}{n} \text{Tr} \Theta \left(\frac{1}{n} \sum_{j=1}^n \frac{R_j R_j^*}{1 + e_j} \right)^{-1} \right| \xrightarrow{a.s.} 0 \quad (.49)$$

where e_1, \dots, e_n are the unique solutions to the following system of equations:

$$e_k = \frac{1}{n} \text{Tr} R_k R_k^* \left(\frac{1}{n} \sum_{j=1}^n \frac{R_j R_j^*}{1 + e_j} \right)^{-1}.$$

Proof. To prove lemma .16, we show first that:

$$\max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} - \frac{1}{n} \text{Tr} R_j R_j^* \Sigma^{-1} \right| \xrightarrow{a.s.} 0$$

From the resolvent identity, we have:

$$\begin{aligned} \max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} - \frac{1}{n} \text{Tr} \Theta \Sigma^{-1} \right| &= \max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} (\Sigma - \Sigma_j) \Sigma_j^{-1} \right| \\ &= \max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma^{-1} R_j \frac{z_j z_j^*}{n} R_j^* \Sigma_j^{-1} \right| \\ &\leq \max_{1 \leq j \leq n} c_N \frac{\|\Theta\| \|R_j\|^2}{\lambda_1(\Sigma) \min_{1 \leq j \leq n} \lambda_1(\Sigma_j^{-1})} \frac{\max_{1 \leq j \leq n} z_j^* z_j}{n^2} \\ &= \max_{1 \leq j \leq n} c_N \frac{\|\Theta\| \|R_j\|^2}{\lambda_1(\Sigma) \min_{1 \leq j \leq n} \lambda_1(\Sigma_j^{-1})} \frac{N + \max_{1 \leq j \leq n} s_j^* s_j}{n^2} \end{aligned}$$

Since there exists $\epsilon > 0$ such that for all large n a.s.

$$\lambda_1(\Sigma) \geq \min_{1 \leq j \leq n} \lambda_1(\Sigma_j) > \epsilon.$$

and $\max_{1 \leq j \leq n} \frac{1}{n} s_j^* s_j$ is almost surely bounded, we have:

$$\max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} - \frac{1}{n} \text{Tr} \Theta \Sigma^{-1} \right| \xrightarrow{a.s.} 0. \quad (.50)$$

Similarly to the previous Lemma, write $w_i = \frac{\sqrt{N} \tilde{w}_i}{\|\tilde{w}_i\|}$, and let $\tilde{z}_i = [s_i^T, \tilde{w}_i^T]^T$. Denote by $\tilde{\Sigma} = \frac{1}{n} \sum_{i=1}^n R_i z_i z_i^* R_i^*$. Then from Lemma .13,

$$\|\Sigma - \tilde{\Sigma}\| \xrightarrow{a.s.} 0.$$

Therefore,

$$\begin{aligned} \max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr } \Theta \Sigma^{-1} - \frac{1}{n} \text{Tr } \Theta \tilde{\Sigma}^{-1} \right| &= \max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr } \Theta \Sigma^{-1} (\tilde{\Sigma} - \Sigma) \tilde{\Sigma}^{-1} \right| \\ &\leq c_N \|\tilde{\Sigma} - \Sigma\| \max_{1 \leq j \leq n} \|\Theta\| \|\Sigma^{-1}\| \|\tilde{\Sigma}^{-1}\| \xrightarrow{a.s.} 0. \end{aligned} \quad (.51)$$

Hence, plugging (.51) into (.50), we get:

$$\max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr } \Theta \Sigma_j^{-1} - \frac{1}{n} \text{Tr } \Theta \tilde{\Sigma}^{-1} \right| \xrightarrow{a.s.} 0.$$

The asymptotic convergence of $\frac{1}{n} \text{Tr } \Theta (\tilde{\Sigma} - z I_N)^{-1}$ has been studied in [19] for $z \in \mathbb{C}_+$. Since the smallest eigenvalue of $\tilde{\Sigma}$ is almost surely away from zero, we can extend the convergence results for $z = 0$ by using the same arguments as those presented in [8, footnote in page 20]. \square

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